4. Lecture notes: Planar graphs

4.1. Basics, classic theorems. A polygonal curve is a curve composed of finitely many line segments.

A drawing of a graph \( G \) is a function \( f : V(G) \cup E(G) \to \mathbb{R}^2 \) s.t.

(a) \( f(v) \in \mathbb{R}^2 \) for every \( v \in V(G) \);
(b) \( f(v) \neq f(v') \) if \( v, v' \in V(G) \) and \( v \neq v' \);
(c) \( f(xy) \) is a polygonal curve connecting \( f(x) \) with \( f(y) \).

A crossing in a drawing of a graph is a common vertex in the images of two edges that is not the image of their common end.

A graph \( G \) is planar if it has a drawing \( f \) without crossings.

A plane graph is a pair \((G, f)\) where \( f \) is a drawing of \( G \) without crossings.

The length, \( \ell(F) \), of a face \( F_i \) in a plane graph \((G, f)\) is the total length of the closed walk(s) bounding \( F_i \).

Restricted Jordan Curve Theorem: A simple closed polygonal curve \( C \) in the plane partitions the plane into exactly two faces each having \( C \) as boundary.

By \( F(G, f) \) we denote the set of faces of the plane graph \((G, f)\).

Proposition 4.1. For each plane graph \((G, f)\),

\[
\sum_{F_i \in F(G, f)} \ell(F_i) = 2|E(G)|.
\]

Proof. By the definition of \( \ell(F_i) \), each edge either contributes 1 to the length of two distinct faces or contributes 2 to the length of one face.

Theorem 4.2 (Euler’s Formula). For every connected plane graph \((G, f)\),

\[|V(G)| - |E(G)| + |F(G, f)| = 2.\]

Corollary 4.3. For \( n \geq 3 \), every simple planar \( n \)-vertex graph \( G \) has at most \( 3n - 6 \) edges. Moreover, if \( G \) is triangle-free, then \( G \) has at most \( 2n - 4 \) edges.

Here Lecture 30 ended.

Corollary 4.4. Graphs \( K_5 \) and \( K_{3,3} \) are not planar.

A Kuratowski graph is a subdivision of \( K_5 \) or \( K_{3,3} \). It follows from Euler’s Formula that neither \( K_5 \) nor \( K_{3,3} \) is planar. Thus every Kuratowski graph is nonplanar. Our goal is to prove the following classic theorem.

Theorem 4.5 (Kuratowski, 1930). A graph \( G \) is planar if and only if \( G \) does not contain a Kuratowski subgraph.

The “only if” part is already proved. Let us prove the “if” part.

Claim 4.6. For every graph \( G \) and any \( xy \in E(G) \), if \( G \) does not contain a Kuratowski subgraph, then \( G/xy \) also doesn’t.
**Proof.** Suppose that \( G/xy \) contains a Kuratowski subgraph \( H \). Let \( z \) be the vertex resulting from contracting \( x \) with \( y \). If \( z \notin V(H) \), then \( H \) is a Kuratowski subgraph of \( G \). If \( z \in V(H) \) but is not a branch vertex of \( H \), then we can obtain a Kuratowski subgraph \( H' \) of \( G \) by replacing \( z \) in \( H \) with either \( x \), or \( y \), or \( \{x, y\} \). The same holds if \( z \) is a branch vertex of \( H \), and at most one edge of \( H \) incident with \( z \) is incident with \( x \) in \( G \). Thus the remaining case is that \( H \) is a subdivision of \( K_5 \) and exactly two edges of \( H \) incident with \( z \) are incident with \( x \) in \( G \) (see Fig. 1 (left)).

![Diagram](image)

**Figure 1**

Then \( G \) contains a subdivision of \( K_{3,3} \) as in Fig. 1 (right). \( \square \)

First, we will prove a stronger statement for 3-connected graphs. A **convex embedding** of a planar graph \( G \) is one in which every edge of \( G \) forms a straight segment and every face (including the outer face) is a convex polygon. Not every planar graph has a convex embedding; for example, \( K_{2,4} \) has not.

**Theorem 4.7** (Tutte). Every 3-connected graph with no Kuratowski subgraph has a convex embedding in the plane with no three vertices on a line.

**Proof.** By induction on \( n := |V(G)| \). If \( n \leq 4 \), then the only 3-connected graph is \( K_4 \), and \( K_4 \) has such embedding.

Suppose the theorem holds for all graphs with at most \( n - 1 \) vertices. Let \( G \) be any \( n \)-vertex 3-connected graph with no Kuratowski subgraph. By Contraction Lemma (7.2.7 in the book), \( G \) has an edge \( xy \) such that \( H := G/xy \) is 3-connected. By Claim \( \square \), \( H \) has no Kuratowski subgraph. So by the IH, \( H \) has a convex embedding in the plane with no three vertices on a line. Fix such an embedding. Let \( z \) be the result of contracting \( xy \) and \( H' \) be obtained from \( H \) by deleting all edges incident with \( z \). Since \( H' - z \) is 2-connected, the face \( C \) of \( H' \) containing \( z \) is a cycle. Let \( x_1, \ldots, x_k \) be the neighbors of \( x \) on \( C \) in cyclic order. If there is some \( i \) such that all neighbors of \( y \) on \( C \) are in the portion of \( C \) between \( x_i \) and \( x_{i+1} \), then we can obtain a convex embedding of \( G \) with no three vertices on a line by placing \( x \) into the position of \( z \) and placing \( y \) very close to \( x \). If this does not happen, then either (a) \( y \) and \( x \) have 3 common neighbors, say \( u, v, w \), or (b) for some \( i < j \), \( y \) has a neighbor \( v \) on \( C \) between \( x_i \) and \( x_j \) (in clockwise order) and a neighbor \( u \) between \( x_j \) and \( x_i \).

In Case (a) we have a \( K_5 \)-subdivision and in Case (b) we have a \( K_{3,3} \)-subdivision. \( \square \)
In order to prove Theorem 4.9, it is now enough to show the following.

**Lemma 4.8.** If $G$ has the fewest vertices among the nonplanar graphs with no Kuratowski subgraphs, then $G$ is 3-connected.

**Proof.** We need the following simple observation:

1. If $F$ is a face in an embedding of a graph $G$ in the plane, then there is an embedding of $G$ in the plane where $F$ the outer face.

If $G$ is disconnected, then by the minimality of $G$, each of its components could be embedded in the plane. The union of these embeddings will be an embedding of $G$. Suppose $G$ has a cut vertex $x$ and $H$ is a component of $G - x$. Let $H_1 = G[V(H) + x]$ and $H_2 = G - H$. By the minimality of $G$, each of $H_1$ and $H_2$ could be embedded in the plane. Then by (**), each of $H_1$ and $H_2$ has an embedding in the plane such that $x$ is on the outer face. Stretching each of these embeddings so that each of the graphs is in one half-plane passing through $x$, we can then glue them into an embedding of $G$.

Suppose now that $G$ is 2-connected and that sets $V_1, V_2 \subset V(G)$ and vertices $x, y$ are such that $V_1 \cup V_2 = V(G)$, $V_1 \cap V_2 = \{x, y\}$ and there are no edges between $V_1 - x - y$ and $V_2 - x - y$. For $i = 1, 2$, let $G_i$ be the graph obtained from $G[V_i]$ by adding edge $xy$. If both $G_1$ and $G_2$ are planar, then by (**), there are their embeddings with edge $xy$ on the outer face. Again, we can stretch these embeddings so that we can glue them along $xy$ and get an embedding of $G$. Thus we may assume that $G_1$ is not planar. By the minimality of $G$, $G_1$ contains a Kuratowski subgraph $H$. Since $G$ does not contain Kuratowski subgraphs, $H$ contains edge $xy$. So we can get a Kuratowski subgraph $H'$ of $G$ from $H$ by replacing $xy$ with an $x, y$-path in $G[V_2]$. Such an $x, y$-path exists, since $G$ is 2-connected and so each of $x$ and $y$ has a neighbor in every component of $G - x - y$. □

Here Lecture 31 ended.

**Theorem 4.9** (Wagner, 1937). A graph $G$ is planar if and only if $G$ does not contain a subgraph contractible to $K_5$ or $K_{3,3}$.

**Proof.** The difficult part by Kuratowski’s Theorem. □

The cycle space, $\mathcal{C}(G)$, of a graph $G$ is the set of characteristic vectors of even subgraphs, i.e. of edge-disjoint unions of cycles in $G$.

The bond space, $\mathcal{B}(G)$, of a graph $G$ is the set of characteristic vectors of edge cuts in $G$.

**Theorem 4.10.** The cycle space and the bond space of a connected $n$-vertex graph $G$ with $m$ edges are binary vector spaces with dimensions $m - n + 1$ and $n - 1$, resp. They are orthogonal complements to each other in $\mathbb{R}^m$.

**Proof.** Check that the sum of char. vectors of even subgraphs (resp. of edge cuts) is again a char. vector of an even subgraph (resp. of an edge cut).

Fix a spanning tree $T$. Each $e \in E(G) - E(T)$ forms a cycle with a part of $T$, and the char. vectors of all these cycles are linearly independent. Thus, $\dim(\mathcal{C}(G)) \geq m - n + 1$.

Fix a vertex $v \in V(G)$. For each $w \in V(G) - v$, let $B_w$ be the edge cut separating $w$ from the rest. The char. vectors of all these cuts are linearly independent. Thus, $\dim(\mathcal{B}(G)) \geq n - 1$.  

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Since each cycle intersects each edge cut in an even number of edges, \( C \perp B \) for any \( C \in \mathcal{C} \) and \( B \in \mathcal{B} \). So by Rank-Nullity Theorem, \( \dim(\mathcal{C}(G)) + \dim(\mathcal{B}(G)) \leq m \). Thus, \( \dim(\mathcal{C}(G)) = m - n + 1 \) and \( \dim(\mathcal{B}(G)) = n - 1 \). \( \square \)

A 2-basis for a linear subspace \( L \) of a space with a given basis \( B \) is a basis of \( L \) s.t. each coordinate is non-zero in at most two vectors of this basis.

**Theorem 4.11** (MacLane). A graph \( G \) is planar iff \( \mathcal{C}(G) \) has a 2-basis.

**Proof.** \((\Rightarrow)\) If \( G \) is not 2-connected, then any basis of \( \mathcal{C}(G) \) is the disjoint union of bases of cycle spaces of its blocks. If \( G \) is planar and 2-connected, and \( f \) is its planar drawing, then the facial cycles of all its bounded faces form a 2-basis for \( \mathcal{C}(G) \).

\((\Leftarrow)\) Suppose \( G \) is not planar. Then it has a subdivision of \( K_5 \) or \( K_{3,3} \).

**Claim 1:** \( \mathcal{C}(K_5) \) has no 2-basis.

**Proof of Claim 1:** By Theorem 31, a basis of \( \mathcal{C}(K_5) \) contains \( m - n + 1 = 10 - 5 + 1 = 6 \) even graphs \( C_1, \ldots, C_6 \). Let \( C_0 = \sum_{i=1}^{6} C_i \). Note that each edge of \( G \) is in at most two of \( C_0, \ldots, C_6 \). Also \( C_0 \neq \emptyset \), since it is a nontrivial sum of basis vectors. But

\[
\sum_{i=0}^{6} |C_i| \geq 7 \cdot 3 = 21 > 2|E(K_5)|.
\]

**Claim 2:** \( \mathcal{C}(K_{3,3}) \) has no 2-basis.

**Proof of Claim 2:** Repeat the proof of Claim 1, but the length of each cycle is now at least 4.

**Claim 3:** The space \( \mathcal{C}(H) \) of a subdivision \( H \) of a graph \( G \) has a 2-basis iff the space \( \mathcal{C}(G) \) has a 2-basis.

**Proof of Claim 3:** Check for subdividing an edge.

**Claim 4:** If \( \mathcal{C}(G) \) has a 2-basis, then for any \( e \in E(G) \), \( \mathcal{C}(G - e) \) has a 2-basis.

**Proof of Claim 4:** If \( e \) is a cut edge, then \( \mathcal{C}(G - e) = \mathcal{C}(G) \). Suppose \( e \) is not. Then \( \dim(\mathcal{C}(G - e)) = \dim(\mathcal{C}(G)) - 1 \). Let \( \mathcal{C} = \{C_1, \ldots, C_k\} \) be a 2-basis of \( \mathcal{C}(G) \).

If \( e \in C_1 \) and to no other \( C_i \), then \( \mathcal{C}' = \mathcal{C} - C_1 \) is a 2-basis of \( \mathcal{C}(G - e) \). If \( e \in C_1 \cap C_2 \), then \( \mathcal{C}' = \mathcal{C} - \{C_1, C_2\} \cup (C_1 + C_2) \) is a 2-basis of \( \mathcal{C}(G - e) \). This proves Claim 4.

The claims together with the fact that \( G \) has a subdivision of \( K_5 \) or \( K_{3,3} \) prove the theorem. \( \square \)

A bond is an edge cut whose edge set does not contain edge sets of other nontrivial edge cuts.

For a multigraph \( G \), a multigraph \( H \) is an abstract dual to \( G \) if there is a bijection \( f : E(G) \to E(H) \) s.t. \( X \subseteq E(G) \) is a cycle in \( G \) \( \iff \) \( f(X) \) is a bond in \( H \).

**Theorem 4.12.** A graph \( G \) is planar iff \( G \) has an abstract dual.

**Proof.** First, observe that \( G \) is planar iff each its block is planar. Also, \( G \) has an abstract dual iff each its block has an abstract dual and the images of edge sets of distinct blocks are disjoint. So, we prove the theorem for 2-connected graphs.

\((\Rightarrow)\) If \( G \) is planar, then its geometric dual is its abstract dual.
(⇐) Suppose $G$ has an abstract dual $H$ (using map $f$). The basis for $B(H)$ constructed in the proof of Theorem ?? is a 2-basis. Since $f$ creates a bijection between $C(G)$ and $B(H)$, $C(G)$ has a 2-basis. □

4.2. Schnyder labelings. Let $(G, f)$ be a triangulation. Then a cell is a bounded face. An angle in a cell $c$ is a pair $(c, v)$ where $v$ is a vertex of this cell, so each cell has 3 angles, and each vertex $v$ is in $d(v)$ angles.

A Schnyder labeling of a triangulation $(G, f)$ is a labeling of the angles in each cell with 1, 2 and 3 such that

1. the angles in each cell are labeled with 1, 2 and 3 in clockwise order, and
2. each interior vertex has angles with each label appearing in clockwise order: first all ones, then all two’s and the all 3’s.

Example!

Observation: Given a Schnyder labeling of a triangulation $(G, f)$, if two cells $abc$ and $abd$ share edge $ab$ and their clockwise orders are $a, c, b$ and $a, b, d$, then the labels at $a$ are distinct, and the labels at $b$ coincide, and differ from the labels at $a$.

This allows us to define an orientation and an edge coloring of $(G, f)$ (see the book). In particular, the outdegree of each internal vertex is 3.

Lemma 4.13. The external vertices can be labeled $v_1, v_2, v_3$ so that for each $1 \leq i \leq 3$ all internal angles at $v_i$ have label $i$.

Proof. Since $G$ has $3n - 9$ internal edges and from each of the $n - 3$ internal vertices start 3 edges, the directed edges only enter the exterior vertices. □

Call an internal edge of a triangulation contractible if its end vertices have only two common neighbors.

Lemma 4.14. If $a$ is an external vertex of a triangulation $(G, f)$ with $|V(G)| \geq 4$, then some internal edge $au$ is contractible.

Proof. Let the neighborhood of $a$ is a cycle $C = x_1x_2 \ldots x_k, x_1$ where $x_1, x_k$ are external (draw a picture!). Choose a shortest chord $x_ix_{i+t}$ of the path $C - x_1x_k$. Then $ax_{i+1}$ is contractible. □

———Here Lecture 33 ended.———

Theorem 4.15. Each triangulation has a Schnyder labeling.

Proof. By induction with contractions (see the book). □

The next lemma shows that all Schnyder labelings appear ”this way”.

Lemma 4.16. Let $L$ be a Schnyder labeling of a triangulation $(G, f)$ with $|V(G)| \geq 4$. Then for each $1 \leq i \leq 3$, $v_i$ has an internal neighbor $u_i$ s.t.

(a) $v_iu_i$ is contractible and
(b) all internal angles at $u_i$ not involving $v_i$ are labeled by $i$. 
Proof. Let the neighborhood of \( v_i \) be a cycle \( C = x_1x_2\ldots x_k, x_1 \) where \( x_1 = v_{i-1}, x_k = v_{i+1} \) (draw a picture!).

Each of \( x_jx_{j+1} \) has an orientation, in particular, \( \overrightarrow{x_1x_2} \) and \( \overrightarrow{x_{k-1}x_k} \). So, there is \( j \) s.t. \( \overrightarrow{x_{j-1}x_j} \) and \( \overrightarrow{x_jx_{j+1}} \). Then we have the 3 edges starting from \( x_j \), so (b) holds.

If \( v_i x_j \) is not contractible, then we may assume there is some \( s \) s.t. \( x_j, x_{j+s} \in \text{E}(G) \). (Pictures!)

By (b), the orientation is \( \overrightarrow{x_{j+s}x_j} \). But then two edges, \( \overrightarrow{x_{j+s}x_j} \) and \( \overrightarrow{x_jx_{j+s}} \), of color \( i \) start from \( x_{j+s} \), a contradiction. \( \square \)

\textbf{Theorem 4.17} (Uniform Angle Lemma). In every Schnyder labeling of a triangulation \( (G, f) \) for each \( 1 \leq i \leq 3 \) and each cycle \( C \) in \( G \), there is an \( i \)-uniform vertex \( x_i \) on \( C_i \), i.e. all angles at \( x_i \) inside \( C \) have label \( i \).

Proof. By induction on \( n = |V(G)| \). If \( C \) visits all external vertices, then O.K. (In particular, \( n > 3 \).)

Otherwise, suppose \( v_1 \notin V(C) \). By Lemma ??, there is \( u_1 \in N(v_1) - v_2 - v_3 \) s.t. contracting \( v_1u_1 \) leads to a smaller triangulation \( G', f' \) with “the same” labeling \( L' \). By minimality, \( C \) has a 1-uniform vertex \( x_1 \).

If \( u_1 \notin V(C) \), then nothing changes at \( x_1 \). If \( u_1 \in V(C) \), then \( C \) visits \( v_1 \) in \( G' \). So, again \( x_1 \) is 1-uniform in \( C \). \( \square \)

\textbf{Theorem 4.18} (Tree Lemma). In every Schnyder labeling of a triangulation \( (G, f) \) for each \( 1 \leq i \leq 3 \) the edges of color \( i \) form an \( (n-2) \)-vertex in-tree \( T_i \) with root \( v_i \). Also, for each internal vertex \( v \), the paths from \( v \) to \( v_i \) in \( T_i \) are internally disjoint for distinct \( i \).

Proof. Let \( T_i \) denote the subgraph \( G \) formed by the edges of color \( i \). Then \( |E(T_i)| = n-3 \) and vertices \( v_{i-1} \) and \( v_{i+1} \) are not in \( T_i \). Suppose first that \( T_i \) has a cycle \( C = x_1x_2\ldots x_kx_1 \). Since only one edge of color \( i \) starts from each \( x_j \), \( C \) is a directed cycle, say \( \overrightarrow{x_jx_{j+1}} \in E(C) \) for each \( j \). But then for each \( j \) label \( i \) is present at the end of each \( \overrightarrow{x_jx_{j+1}} \) inside \( C \), contradicting Theorem ??.

Thus, \( T_i \) has \( n-2 \) vertices, \( n-3 \) edges and no cycles. So, it is a tree. Since no vertex apart from \( v_i \) is a sink in \( T_i \), the tree is an in-tree with root \( v_i \).

Suppose now that for \( u \neq v \) there are \( u, v \)-paths in both \( T_1 \) and \( T_2 \). Choose such \( u \) and \( v \) so that the total length of the paths is minimum. Then these paths, say \( P_1 \) and \( P_2 \) form a cycle, say \( C \). Note that at each internal vertex of \( P_1 \) there is an angle of color 1 and an angle of another color. The same for \( P_2 \) (with 1 switched to 2). Thus uniform vertices can be only \( u \) and \( v \), but Theorem ?? says there are 3 such vertices, a contradiction. \( \square \)

Here Lecture 34 ended.

Lecture 35 was presented by Mina Nahvi.

Let \( P_i(v) \) denote the \( v, v_i \)-path in \( T_i \).

Let \( R_i(v) \) denote the region enclosed by \( P_{i-1}(v), P_{i+1}(v) \) and edge \( v_{i-1}v_{i+1} \).

\textbf{Lemma 4.19.} Let \( 1 \leq i \leq 3 \). If \( u \) and \( v \) are distinct internal vertices in a triangulation \( (G, f) \) and in a Schnyder labeling \( L \) of \( (G, f) \), \( u \in R_i(v) \), then \( R_i(u) \) is properly contained in \( R_i(v) \).
Proof. We may assume that $i = 1$. It is enough to show that neither of $P_2(u)$ and $P_3(u)$ has a vertex outside $R_1(v)$.

Consider first $P_2(u)$. If it comes to $P_2(v)$, then it must follow $P_2(v)$ till the very end. Suppose $P_2(u)$ hits $P_3(v) - P_2(v)$ at $w$. Then $w \neq v$, so there are edges $\overrightarrow{w'w}$ and $\overrightarrow{ww''}$ on $P_3(v)$ (of color 3). But then all edges of color 2 must hit $w$ from the right, i.e. from $R_2(v)$, and the unique edge of color 2 leaving $w$ goes to $R_1(v)$.

Symmetrically, if $P_3(u)$ comes to $P_3(v)$, then it then follows $P_3(v)$ till the very end, and every edge of color 3 leaving $P_2(v)$ goes to $R_1(v)$. □

A digression.

A barycentric representation of $G$ is an injection $\phi : V(G) \to \mathbb{R}^3$ s.t.

(i) the coordinates $v_1, v_2, v_3$ of $\phi(v)$ are nonnegative and $v_1 + v_2 + v_3 = 1$ for each $v \in V(G)$, and

(ii) if $uv \in E(G)$ and $w \in V(G) - u - v$, then $w_i > \max\{u_i, v_i\}$ for some $i \in [3]$.

Lemma 4.20. If $\phi$ is a barycentric representation of $G$, then drawing the edges of $G$ as straight segments connecting the images of the vertices yields a planar drawing of $G$.

Proof. Recall that $\phi$ is an injection. Consider any two edges $uv$ and $wz$ with all 4 vertices distinct. By Part (ii) of the definition, there are indices $i, j, h, k$ s.t.

$$u_i > \max\{w_i, z_i\}, v_j > \max\{w_j, z_j\}, w_h > \max\{u_h, v_h\}, z_k > \max\{u_k, v_k\}.$$ 

By definition, $\{i, j\} \cap \{h, k\} = \emptyset$. By pigeonhole and symmetry, we may assume $i = j$. Then there is $\alpha$ s.t. both $u$ and $v$ are above the line $x_i = \alpha$, and both $w$ and $z$ are below it.

We also need to exclude the situation when say edge $uv$ contains edge $wz$. If this would happen then (ii) would not hold. □

Given a Schnyder labeling $L$ of a triangulation $(G, f)$, for each internal $v$ let $r_i(v)$ denote the number of cells in $R_i(v)$. Also, for an external vertex $v_i$, let $r_i(v_i) = 2n - 5$ and $r_j(v_i) = 0$ when $j \neq i$. Then $r_1(v) + r_2(v) + r_3(v) = 2n - 5$ for all $v \in V(G)$. Now define

$$\phi(v) = \left( \frac{r_1(v)}{2n - 5}, \frac{r_2(v)}{2n - 5}, \frac{r_3(v)}{2n - 5} \right) \quad \forall v \in V(G). \tag{2}$$

Theorem 4.21. The function $\phi$ defined by (2) is a barycentric representation of $G$.

Proof. Part (i) of the definition is clear. Suppose $w$ is not an end of edge $uv$. If $w$ is external, then (ii) is obvious. Suppose $w$ is internal. Then $u$ is in some $R_i(w)$ and $v$ is in some $R_j(w)$. Since they are adjacent, either they both are in $R_i(w)$ or they both are in $R_j(w)$, say they both are in $R_i(w)$. Then $w_i > \max\{u_i, v_i\}$. □

So, each planar graph has a straight-line embedding into the grid points of the triangle with corners $(0, 0)$, $(2n - 5, 0)$ and $(0, 2n - 5)$.

To shrink the size of the triangle, here is a refinement. We will closer follow the book notation for the homework.
A weak barycentric representation of $G$ is an injection $\phi : V(G) \to \mathbb{R}^3$ s.t.

1. the coordinates $v_1, v_2, v_3$ of $\phi(v)$ are nonnegative and $v_1 + v_2 + v_3 = 1$ for each $v \in V(G)$,

2. if $xy \in E(G)$ and $z \in V(G) - x - y$, then for some $k \in [3]$ vectors $(x_k, x_{k+1})$ and $(y_k, y_{k+1})$ are lexicographically less than $(z_k, z_{k+1})$.

For an internal vertex $v$ of a triangulation $(G, f)$ with a Schnyder labeling $L$, let $v_i'$ denote the number of vertices in $R_i(v) - P_{i-1}(v)$. Then $v_1' + v_2' + v_3' = n - 1$. For an external vertex $v_j$, $R_j(v_j)$ has $n$ vertices, $P_j(v_j)$ has one vertex and each of $P_{j-1}(v_j), P_{j+1}(v_j)$ has two vertices. So, let $(v_j)_j' = n - 2, (v_j)_{j+1}' = 1$, and $(v_j)_{j-1}' = 0$.

**Two Lemmas and Theorem in HW5.**

### 4.3. Small separators in planar graphs.

An $(m, \alpha)$-separation of $G$ is a partition $V(G) = A \cup B \cup C$ s.t

(a) $|C| \leq m$,  
(b) $G - C$ has no edges between $A$ and $B$, and  
(c) $|A|, |B| \leq \alpha |V(G)|$.

A class $\mathcal{F}$ of graphs is an $f$-separator with shrink factor $\alpha$ if each $G \in \mathcal{F}$ has an $(f(|V(G)|), \alpha)$-separation.

In general, for each $\epsilon > 0$ there is $c_\epsilon > 0$ s.t. for almost all $G$ with $(2 + \epsilon)k$ vertices and $c_\epsilon k$ edges deleting any $k$ vertices results in a graph with a component with $\geq k$ vertices.

**Lemma 4.22.** Let $(G, f)$ be a near-triangulation with a 2-coloring of vertices with red and blue. If the outer cycle $C$ has red vertices $x, y$, then $G$ has either a red $x, y$-path or a blue path connecting the components of $C - x - y$.

**Proof.** Consider the set $A$ of the red vertices reachable from $x$ via red paths. Consider its neighborhood. Use triangualtion (in class). □

**Lemma 4.23.** Let $(G, f)$ be a near-triangulation with the outer cycle $C = v_0, v_1, \ldots, v_{2k-1}, v_1$. If $G$ has no $v_0, v_k$-path of length at most $k - 1$, then there are $k - 1$ disjoint paths $P_1, \ldots, P_{k-1}$ where $P_i$ connects $v_i$ with $v_{2k-i}$.

**Proof.** Let $S$ be a smallest set separating $X = \{v_1, \ldots, v_{k-1}\}$ from $Y = \{v_{k+1}, \ldots, v_{2k-1}\}$ in $G - v_0 - v_k$. Let $\text{Red} = S \cup \{v_0, v_k\}$. By definition, $G$ has no blue $X,Y$-path. Then by Lemma ??, $G$ has a red $v_0, v_k$-path. But this path has at least $k - 1$ internal vertices, so $|S| \geq k - 1$. By Menger (Pym), there are $k - 1$ disjoint $X,Y$-paths. They form the linkage we promised, since $(G, f)$ is plane. □

**Theorem 4.24** (Lipton and Tarjan). For each $n \geq 1$ each $n$-vertex planar graph has a $(2\sqrt{2n}, 2/3)$-separation.

**Proof.** So we prove the theorem for triangulations by induction on $n$. If $n < 30$, then simply delete any $\lfloor \sqrt{8n} \rfloor$ vertices. Let $k = \lfloor \sqrt{2n} \rfloor$.

**Here Lecture 37 ended.**

Define $C^+$ and $C^-$, $c^+ = |C^+|$ and $c^- = |C^-|$.

Among the cycles $C$ with $c^+ \geq 2n/3$ and $|C| \leq 2k$, choose one with the minimum $c^- - c^+$. Here Lecture 37 ended.
If \( c^- \leq 2n/3 \), we are done. Suppose \( c^- > 2n/3 \). Let \( D = G[C^- \cup V(C)] \). For \( u, v \in V(C) \), let \( c(u, v) \) (resp., \( d(u, v) \)) be the distance between \( u \) and \( v \) in \( C \) (resp., in \( D \)). Since \( C \subset D \), \( c(u, v) \leq d(u, v) \).

**Claim 1:** \( d(u, v) = c(u, v) \) for all \( u, v \in V(C) \).

Indeed, if not, choose a wrong pair \((u, v)\) with minimum \( d(u, v) \). Let \( P \) be a shortest \( u, v \)-path in \( G \). By minimality, \( V(P) \cap V(C) = \{u, v\} \). \( P \) forms with \( C \) two cycles, \( C_1 \) and \( C_2 \). Suppose \( c^-_1 \geq c^-_2 \). By construction, \( |C_1| \leq |C| \leq 2k \). Now,

\[
|C| \leq 2k.
\]

Thus, \( c^-_1 \leq \frac{2n}{3} \), contradicting the minimality of \( c^- = c^+ \).

**Claim 2:** \( |C| = 2k \).

If shorter, we can make \( c^- \) smaller (using Claim 1).

So, let \( C = v_0, v_1, \ldots, v_{2k-1}, v_1 \). By Claim 1 and Lemma ??, there are disjoint paths \( P_1, \ldots, P_{k-1} \), where \( P_i \) is a \( v_i, v_{2k-i-1} \)-path. Again by Claim 1, \( |V(P_i)| \geq 1 + \min\{2i, 2(k-i)\} \).

Hence

\[
|V(G)| > |V(D)| \geq (1+3+\ldots+[((k+1)/2)^2])+(1+3+\ldots+[((k+1)/2)^2]) \geq \frac{(k+1)^2}{2} > n,
\]

a contradiction. \( \square \)

Applications.

4.4. Discharging for planar graphs.

Examples of discharging, versions of Euler’s Formula. The FCT.

A normal plane map is a connected plane multigraph whose vertex degrees and face lengths all are at least 3.

Lebesgue (1940) proved that each 3-connected plane graph \((G, f)\) with \( \delta(G) \geq 5 \) has a 3-face \((a, b, c)\) with \( d(a) + d(b) + d(c) \leq 19 \). Kotzig (1963) improved 19 to 18 and in 1979 conjectured 17. An example of 17 is obtained from the dodecahedron by inserting a vertex into each face.

**Theorem 4.25** (Borodin, 1989). Every normal plane map \((G, f)\) with \( \delta(G) \geq 5 \) has a 3-face \((a, b, c)\) with \( d(a) + d(b) + d(c) \leq 17 \).

---

**Proof.** For a given \( n \), consider an edge maximal counter-example \((G, f)\).

**Claim:** For each \( 4^+\)-face \( F \), \( d(v) = 5 \) for every \( v \in F \) (by the maximality of \((G, f)\)).

(Here normal maps are used.)

Define the initial charge: \( ch(v) = d(v) - 6 \) for each \( v \in V(G) \) and \( ch(\alpha) = 2d(\alpha) - 6 \) for each face of \((G, f)\).

**Discharging:**

(R1) Each \( 4^+\)-face gives 1/2 to each its vertex.

(R2) Each 7-vertex gives 1/3 to each its 5-neighbor.
(R3) Each 8+-vertex gives 1/4 to each incident 3-face and then each such 3-face shares the obtained surplus among its 5-vertices.

We will prove that the new charge $ch^*$ is nonnegative for each vertex and face.

First, look at faces. Each 3-face has initial charge 0 and gives out only a surplus. For $k \geq 4$ any $k$-face $\alpha$ gives out by (R1) exactly $k/2$ and remains with the charge $(2k - 6) - k/2 = \frac{3}{2}(k - 4) \geq 0$.

Now, look at vertices. For $k \geq 8$ any $k$-vertex $v$ gives out by (R3) at most $k/4$ and remains with the charge $k - 6 - k/4 = \frac{3}{4}(k - 8) \geq 0$.

Suppose $d(v) = 7$. By the claim, all faces containing $v$ are triangles. Hence no two consecutive neighbors are 5-vertices. It follows that has at most 3 5-neighbors. Thus, $ch^*(v) \geq 7 - 6 - 3(1/3) = 0$.

If $d(v) = 6$, then $v$ keeps its original charge of 0.

Finally, suppose $d(v) = 5$ and the neighbors of $v$ are $x_1, \ldots, x_5$ in clockwise order. At the start, its charge is −1. If $v$ is incident with at least two 4+-faces, then by (R1) it will get from them $2(1/2) = 1$. Let now $v$ be incident with exactly one 4+-face $\alpha$ and let $x_1, v, x_5$ be a part of the boundary of $\alpha$ (see the picture in class). By the claim, $d(x_1) = d(x_5) = 5$. Then $d(x_2) \geq 8$ and $d(x_4) \geq 8$, so $v$ will get from them by (R3) at least $4((1/4)/2) = 1/2$ and get 1/2 from $\alpha$.

Now, assume all faces incident to $v$ are 3-faces. Then at most two neighbors of $v$ are 5-vertices. If exactly two, then the remaining neighbors are 8+-vertices that together will give 1 to $v$ (see pictures in class). Suppose only $x_1$ is the 5-neighbor of $v$. Then $x_2$ and $x_5$ are 8+-vertices and one of $x_3, x_4$ is a 7+-neighbor, say $d(x_3) \geq 7$. Then $x_5$ gives to $v$ 1/8 via face $vx_5x_1$ and 1/4 via face $vx_5x_4$. Similarly, $x_1$ gives to $v$ 3/8. Vertex $x_3$ gives to $v$ 1/3 if $d(x_3) = 7$ and 1/2 if $d(x_3) \geq 8$.

The last subcase of the last case is that $v$ has no 5-neighbors. Then it has at least 3 7+-neighbors, and each of them gives to $v$ at least 1/3. \[\square\]

Recall:

**Theorem 4.26** (Grötzsch). Every planar graph with no triangles is 3-colorable.

**Conjecture (Steinberg, 1976).** Every planar graph with no 4-cycles and 5-cycles is 3-colorable.

**Question (Erdős, 1993).** Does there exists $k \geq 5$ such that every planar graph with no cycles of length 4, 5, \ldots, $k$ is 3-colorable?

Abbott and Zhou: $k = 11$ works.


Voigt (2005): Not for list coloring.

Cohen, Addad, Hebdige, Král, Li and Salgado (2017): $k = 5$ DOES NOT WORK.

We will prove that $k = 9$ works.

**Lemma 4.27** (Borodin (1996)). Every plane graph $(G, f)$ with $\delta(G) \geq 3$ has

(i) two 3-faces with a common edge, or
(ii) a \( j \)-face for some \( 4 \leq j \leq 9 \), or
(iii) a 10-face with all vertices of degree 3.

**Proof.** Suppose none of (a)–(c) holds. Apply the balanced Euler’s Formula:

\[
\sum_{v \in V(G)} (d(v) - 4) + \sum_{\alpha \in F(G,f)} (d(\alpha) - 4) = -8.
\]

For each \( x \in V(G) \cup F(G,f) \), the initial charge is \( ch(x) = d(x) - 4 \). The discharging rules are:

(R1) Each 3-face gets 1/3 from each incident vertex.
(R2) Each non-3-face \( \alpha \) (\( d(\alpha) \geq 10 \)) gives:
(a) 2/3 to each incident 3-vertex incident to a 3-face,
(b) 1/3 to each other incident 3-vertex,
(c) 1/3 to each incident 4-vertex on two 3-faces,
(d) 1/3 to each incident 4-vertex that has a 3-face opposite to \( \alpha \).

Let us check that the new charge \( ch^*(x) \) is nonnegative for each \( x \in V(G) \cup F(G,f) \) (which would be a contradiction). Consider all cases for \( x \):

(A) \( x \) is a 3-face. By (R1), \( ch^*(x) = (3 - 4) + 3(1/3) = 0 \).

(B) \( x \) is a 3-vertex. If \( x \) is in no 3-faces, then by (R2)(b), \( ch^*(x) = (3 - 4) + 3(1/3) = 0 \). If there is an incident 3-face (then only one!), then \( x \) gives to it 1/3 by (R1), but gets 2(2/3) = 4/3 by (R2)(a).

(C) \( x \) is a 4-vertex. If \( x \) is in no 3-faces, then \( ch^*(x) = ch(x) = 0 \). If there is exactly one incident 3-face, then \( x \) gives to it 1/3 by (R1), but gets 1/3 by (R2)(b).

(D) \( x \) is a \( j \)-vertex for some \( j \geq 5 \). Then \( x \) belongs to at most \( \lfloor j/2 \rfloor \) 3-faces, so by (R1), \( ch^*(x) \geq (j - 4) - \lfloor j/2 \rfloor (1/3) > 0 \).

(E) \( x \) is a \( j \)-face for some \( j \geq 10 \), say \( x = v_i, v_2, \ldots, v_j, v_1 \). If \( x \) gives 2/3 to some \( v_i \), then \( d(v_i) = 3 \) and \( v_i \) belongs to a 3-face \( \alpha \).

(3) \( x \) shares exactly one edge with \( x \).

Thus when \( j \) is odd, \( x \) cannot give 2/3 to each its vertex. Hence \( ch^*(x) \geq (j - 4) - j(2/3) \) when \( j \) is even and \( ch^*(x) \geq (j - 4) - j(2/3) + 1/3 \) when \( j \) is odd. This is nonnegative for all \( j \geq 11 \).

Let \( j = 10 \) and exactly \( i \) vertices of \( x \) be 3-vertices on 3-faces. If \( i = 10 \), then Part (iii) of the lemma holds. If \( i \leq 8 \), then \( ch^*(x) \geq (10 - 4) - 8(2/3) - 2(1/3) = 0 \). By (??), \( i \neq 9 \).

**Theorem 4.28** (Borodin, Sanders and Zhao). Every planar graph with no cycles of length 4, 5, \ldots, \( k \) is 3-colorable.

**Proof.** If not, then there is a counter-example \((G, f)\) with the fewest vertices. By minimality, \( G \) is 4-critical, so it is 2-connected and \( \delta(G) \geq 3 \). By Lemma ??, \( G \) has a 10-cycle \( C \) with \( |C| = 10 \). Since \( G \) is 4-critical, \( G - V(C) \) has a 3-coloring \( \phi \). Extend it to \( C \) (in class). \( \square \)