4. Lecture notes: Planar graphs

4.1. **Basics, classic theorems.** A polygonal curve is a curve composed of finitely many line segments.

A drawing of a graph G is a function $f : V(G) \cup E(G) \to \mathbf{R^2}$ s.t.
(a) $f(v) \in \mathbf{R}^2$ for every $v \in V(G)$;
(b) $f(v) \neq f(v')$ if $v, v' \in V(G)$ and $v \neq v'$;
(c) $f(xy)$ is a polygonal curve connecting $f(x)$ with $f(y)$.

A *crossing* in a drawing of a graph is a common vertex in the images of two edges that is not the image of their common end.

A graph G is *planar* if it has a drawing f without crossings.

A *plane graph* is a pair (G, f) where f is a drawing of G without crossings.

A **face** of a plane graph (G, f) is a connected component of $\mathbf{R^2} - f(V(G) \cup E(G))$.

The *length*, $\ell(F_i)$, of a face F_i in a plane graph (G, f) is the total length of the closed walk(s) bounding F_i.

Restricted Jordan Curve Theorem: *A simple closed polygonal curve C in the plane partitions the plane into exactly two faces each having C as boundary.*

By $F(G, f)$ we denote the set of faces of the plane graph (G, f).

Proposition 4.1. *For each plane graph (G, f),*

$$\sum_{F_i \in F(G,f)} \ell(F_i) = 2|E(G)|. \tag{1}$$

Proof. By the definition of $\ell(F_i)$, each edge either contributes 1 to the length of two distinct faces or contributes 2 to the length of one face.

Theorem 4.2 (Euler's Formula)**.** *For every connected plane graph (G, f),*

$$|V(G)| - |E(G)| + |F(G, f)| = 2.$$

Corollary 4.3. *For $n \geq 3$, every simple planar n-vertex graph G has at most $3n - 6$ edges. Moreover, if G is triangle-free, then G has at most $2n - 4$ edges.*

———————————— **Here Lecture 30 ended.**

Corollary 4.4. *Graphs K_5 and $K_{3,3}$ are not planar.*

A *Kuratowski graph* is a subdivision of K_5 or $K_{3,3}$. It follows from Euler's Formula that neither K_5 nor $K_{3,3}$ is planar. Thus every Kuratowski graph is nonplanar. Our goal is to prove the following classic theorem.

Theorem 4.5 (Kuratowski, 1930)**.** *A graph G is planar if and only if G does not contain a Kuratowski subgraph.*

The "only if" part is already proved. Let us prove the "if" part.

Claim 4.6. *For every graph G and any $xy \in E(G)$, if G does not contain a Kuratowski subgraph, then G/xy also doesn't.*

Proof. Suppose that G/xy contains a Kuratowski subgraph H. Let z be the vertex resulting from contracting x with y. If $z \notin V(H)$, then H is a Kuratowski subgraph of G. If $z \in V(H)$ but is not a branch vertex of H, then we can obtain a Kuratowski subgraph H' of G by replacing z in H with either x, or y, or $\{x, y\}$. The same holds if z is a branch vertex of H, and at most one edge of H incident with z is incident with x in G. Thus the remaining case is that H is a subdivision of K_5 and exactly two edges of H incident with z are incident with x in G (see Fig. 1 (left)).

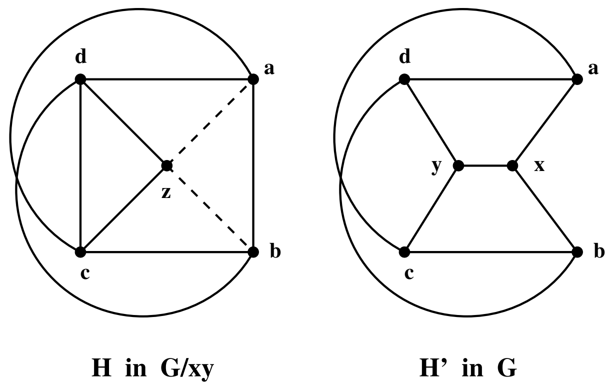

FIGURE 1

Then G contains a subdivision of $K_{3,3}$ as in Fig. 1 (right). $\square$

First, we will prove a stronger statement for 3-connected graphs. A *convex embedding* of a planar graph G is one in which every edge of G forms a straight segment and every face (including the outer face) is a convex polygon. Not every planar graph has a convex embedding; for example, $K_{2,4}$ has not.

Theorem 4.7 (Tutte)**.** *Every 3-connected graph with no Kuratowski subgraph has a convex embedding in the plane with no three vertices on a line.*

Proof. By induction on $n := |V(G)|$. If $n \leq 4$, then the only 3-connected graph is K_4, and K_4 has such embedding.

Suppose the theorem holds for all graphs with at most $n-1$ vertices. Let G be any n-vertex 3-connected graph with no Kuratowski subgraph. By Contraction Lemma (7.2.7 in the book), G has an edge xy such that $H := G/xy$ is 3-connected. By Claim **??**, H has no Kuratowski subgraph. So by the IH, H has a convex embedding in the plane with no three vertices on a line. Fix such an embedding. Let z be the result of contracting xy and H' be obtained from H by deleting all edges incident with z. Since $H' - z$ is 2-connected, the face C of H' containing z is a cycle. Let $x_1, \ldots, x_k$ be the neighbors of x on C in cyclic order. If there is some i such that all neighbors of y on C are in the portion of C between x_i and x_{i+1}, then we can obtain a convex embedding of G with no three vertices on a line by placing x into the position of z and placing y very close to x. If this does not happen, then either (a) y and x have 3 common neighbors, say u, v, w, or (b) for some $i < j$, y has a neighbor v on C between x_i and x_j (in clockwise order) and a neighbor u between x_j and x_i.

In Case (a) we have a K_5-subdivision and in Case (b) we have a $K_{3,3}$-subdivision. $\square$

In order to prove Theorem **??**, it is now enough to show the following.

Lemma 4.8. *If G has the fewest vertices among the nonplanar graphs with no Kuratowski subgraphs, then G is 3-connected.*

Proof. We need the following simple observation:
(**) *If F is a face in an embedding of a graph G in the plane, then there is an embedding of G in the plane where F the outer face.*

If G is disconnected, then by the minimality of G, each of its components could be embedded in the plane. The union of these embeddings will be an embedding of G. Suppose G has a cut vertex x and H is a component of $G-x$. Let $H_1 = G[V(H)+x]$ and $H_2 = G-H$. By the minimality of G, each of H_1 and H_2 could be embedded in the plane. Then by (**) each of H_1 and H_2 has an embedding in the plane such that x is on the outer face. Stretching each of these embeddings so that each of the graphs is in one half-plane passing through x, we can then glue them into an embedding of G.

Suppose now that G is 2-connected and that sets $V_1, V_2 \subset V(G)$ and vertices x, y are such that $V_1 \cup V_2 = V(G)$, $V_1 \cap V_2 = \{x, y\}$ and there are no edges between $V_1 - x - y$ and $V_2 - x - y$. For $i = 1, 2$, let G_i be the graph obtained from $G[V_i]$ by adding edge xy. If both G_1 and G_2 are planar, then by (**), there are their embeddings with edge xy on the outer face. Again, we can stretch these embeddings so that we can glue them along xy and get an embedding of G. Thus we may assume that G_1 is not planar. By the minimality of G, G_1 contains a Kuratowski subgraph H. Since G does not contain Kuratowski subgraphs, H contains edge xy. So we can get a Kuratowski subgraph H' of G from H be replacing xy with an x, y-path in $G[V_2]$. Such an x, y-path exists, since G is 2-connected and so each of x and y has a neighbor in every component of $G - x - y$. $\square$

———————— **Here Lecture 31 ended.**

Theorem 4.9 (Wagner, 1937)**.** *A graph G is planar if and only if G does not contain a subgraph contractible to K_5 or $K_{3,3}$.*

Proof. The difficult part by Kuratowski's Theorem. $\square$

The *cycle space*, $\mathcal{C}(G)$, of a graph G is the set of characteristic vectors of *even subgraphs*, i.e. of edge-disjoint unions of cycles in G.

The *bond space*, $\mathcal{B}(G)$, of a graph G is the set of characteristic vectors of edge cuts in G.

Theorem 4.10. *The cycle space and the bond space of a connected n-vertex graph G with m edges are binary vector spaces with dimensions $m-n+1$ and $n-1$, resp. They are orthogonal complements to each other in $\mathbf{R}^m$.*

Proof. Check that the sum of char. vectors of even subgraphs (resp. of edge cuts) is again a char. vector of an even subgraph (resp. of an edge cut).

Fix a spanning tree T. Each $e \in E(G) - E(T)$ forms a cycle with a part of T, and the char. vectors of all these cycles are linearly independent. Thus, $\dim(\mathcal{C}(G)) \geq m - n + 1$.

Fix a vertex $v \in V(G)$. For each $w \in V(G) - v$, let B_w be the edge cut separating w from the rest. The char. vectors of all these cuts are linearly independent. Thus, $\dim(\mathcal{B}(G)) \geq n - 1$.

Since each cycle intersects each edge cut in an even number of edges, $C \perp B$ for any $C \in \mathcal{C}$ and $B \in \mathcal{B}$. So by Rank-Nullity Theorem, $\dim(\mathcal{C}(G)) + \dim(\mathcal{B}(G))) \leq m$. Thus, $\dim(\mathcal{C}(G)) = m - n + 1$ and $\dim(\mathcal{B}(G)) = n - 1$. $\square$

A 2-*basis* for a linear subspace L of a space with a given basis B is a basis of L s.t. each coordinate is non-zero in at most two vectors of this basis.

Theorem 4.11 (MacLane)**.** *A graph G is planar iff $\mathcal{C}(G)$ has a 2-basis.*

Proof. ($\Rightarrow$) If G is not 2-connected, then any basis of $\mathcal{C}(G)$ is the disjoint union of bases of cycle spaces of its blocks. If G is planar and 2-connected, and f is its planar drawing, then the facial cycles of all its bounded faces form a 2-basis for $\mathcal{C}(G)$.

($\Leftarrow$) Suppose G is not planar. Then it has a subdivision of K_5 or $K_{3,3}$.

Claim 1: *$\mathcal{C}(K_5)$ has no 2-basis.*

Proof of Claim 1: By Theorem **??**, a basis of $\mathcal{C}(K_5)$ contains $m - n + 1 = 10 - 5 + 1 = 6$ even graphs $C_1, \ldots, C_6$. Let $C_0 = \sum_{i=1}^{6} C_i$. Note that each edge of G is in at most two of $C_0, \ldots, C_6$. Also $C_0 \neq \emptyset$, since it is a nontrivial sum of basis vectors. But

$$\sum_{i=0}^{6} |C_i| \geq 7 \cdot 3 = 21 > 2|E(K_5)|.$$

Claim 2: *$\mathcal{C}(K_{3,3})$ has no 2-basis.*

Proof of Claim 2: Repeat the proof of Claim 1, but the length of each cycle is now at least 4.

Claim 3: *The space $\mathcal{C}(H)$ of a subdivision H of a graph G has a 2-basis iff the space $\mathcal{C}(G)$ has a 2-basis.*

Proof of Claim 3: Check for subdividing an edge.

Claim 4: *If $\mathcal{C}(G)$ has a 2-basis, then for any $e \in E(G)$, $\mathcal{C}(G - e)$ has a 2-basis.*

———————————— **Here Lecture 32 ended.**

Proof of Claim 4: If e is a cut edge, then $\mathcal{C}(G - e) = \mathcal{C}(G)$. Suppose e is not. Then $\dim(\mathcal{C}(G - e)) = \dim(\mathcal{C}(G)) - 1$. Let $\mathbf{C} = \{C_1, \ldots, C_k\}$ be a 2-basis of $\mathcal{C}(G)$.

If $e \in C_1$ and to no other C_i, then $\mathbf{C}' = \mathbf{C} - C_1$ is a 2-basis of $\mathcal{C}(G - e)$. If $e \in C_1 \cap C_2$, then $\mathbf{C}' = \mathbf{C} - \{C_1, C_2\} \cup (C_1 + C_2)$ is a 2-basis of $\mathcal{C}(G - e)$. This proves Claim 4.

The claims together with the fact that G has a subdivision of K_5 or $K_{3,3}$ prove the theorem. $\square$

A *bond* is an edge cut whose edge set does not contain edge sets of other nontrivial edge cuts.

For a multigraph G, a multigraph H is an *abstract dual* to G if there is a bijection $f : E(G) \to E(H)$ s.t.

$X \subseteq E(G)$ is a cycle in G $\Leftrightarrow$ $f(X)$ is a bond in H.

Theorem 4.12. *A graph G is planar iff G has an abstract dual.*

Proof. First, observe that G is planar iff each its block is planar. Also, G has an abstract dual iff each its block has an abstract dual and the images of edge sets of distinct blocks are disjoint. So, we prove the theorem for 2-connected graphs.

($\Rightarrow$) If G is planar, then its geometric dual is its abstract dual.

(⇐) Suppose G has an abstract dual H (using map f). The basis for $\mathcal{B}(H)$ constructed in the proof of Theorem **??** is a 2-basis. Since f creates a bijection between $\mathcal{C}(G)$ and $\mathcal{B}(H)$, $\mathcal{C}(G)$ has a 2-basis. □

4.2. **Schnyder labelings.** Let (G, f) be a triangulation. Then a *cell* is a bounded face. An *angle* in a cell c is a pair (c, v) where v is a vertex of this cell, so each cell has 3 angles, and each vertex v is in $d(v)$ angles.

A **Schnyder labeling** of a triangulation (G, f) is a labeling of the angles in each cell with $1, 2$ and 3 such that

(1) the angles in each cell are labeled with 1,2 and 3 in clockwise order, and

(2) each interior vertex has angles with each label appearing in clockwise order: first all ones, then all two's and the all 3's.

Example!

Observation: *Given a Schnyder labeling of a triangulation (G, f), if two cells abc and abd share edge ab and their clockwise orders are a, c, b and a, b, d, then the labels at a are distinct, and the labels at b coincide, and differ from the labels at a.*

This allows us to define an orientation and an edge coloring of (G, f) (see the book). In particular, the outdegree of each internal vertex is 3.

Lemma 4.13. *The external vertices can be labeled v_1, v_2, v_3 so that for each $1 \leq i \leq 3$ all internal angles at v_i have label i.*

Proof. Since G has $3n - 9$ internal edges and from each of the $n - 3$ internal vertices start 3 edges, the directed edges only enter the exterior vertices. □

Call an internal edge of a triangulation *contractible* if its end vertices have only two common neighbors.

Lemma 4.14. *If a is an external vertex of a triangulation (G, f) with $|V(G)| \geq 4$, then some internal edge au is contractible.*

Proof. Let the neighborhood of a is a cycle $C = x_1x_2 \ldots x_k, x_1$ where x_1, x_k are external (draw a picture!). Choose a shortest chord x_ix_{i+t} of the path $C - x_1x_k$. Then ax_{i+1} is contractible. □

——————————— **Here Lecture 33 ended.**

Theorem 4.15. *Each triangulation has a Schnyder labeling.*

Proof. By induction with contractions (see the book). □

The next lemma shows that all Schnyder labelings appear "this way".

Lemma 4.16. *Let L be a Schnyder labeling of a triangulation (G, f) with $|V(G)| \geq 4$. Then for each $1 \leq i \leq 3$, v_i has an internal neighbor u_i s.t.*

(a) v_iu_i is contractible and

(b) all internal angles at u_i not involving v_i are labeled by i.

Proof. Let the neighborhood of v_i be a cycle $C = x_1x_2\dots x_k, x_1$ where $x_1 = v_{i-1}, x_k = v_{i+1}$ (draw a picture!).

Each of x_jx_{j+1} has an orientation, in particular, $\overleftarrow{x_1x_2}$ and $\overrightarrow{x_{k-1}x_k}$. So, there is j s.t. $\overleftarrow{x_{j-1}x_j}$ and $\overrightarrow{x_jx_{j+1}}$. Then we have the 3 edges starting from x_j, so (b) holds.

If v_ix_j is not contractible, then we may assume there is some s s.t. $x_j, x_{j+s} \in E(G)$. (Pictures!)

By (b), the orientation is $\overrightarrow{x_{j+s}x_j}$. But then two edges, $\overrightarrow{x_{j+s}x_j}$ and $\overrightarrow{x_{j+s}v_i}$, of color i start from x_{j+s}, a contradiction. $\square$

Theorem 4.17 (Uniform Angle Lemma)**.** *In every Schnyder labeling of a triangulation (G, f) for each $1 \le i \le 3$ and each cycle C in G, there is an i-**uniform vertex** x_i on C_i, i.e. all angles at x_i inside C have label i.*

Proof. By induction on $n = |V(G)|$. If C visits all external vertices, then O.K. (In particular, $n > 3$.)

Otherwise, suppose $v_1 \notin V(C)$. By Lemma **??**, there is $u_1 \in N(v_1) - v_2 - v_3$ s.t. contracting v_1u_1 leads to a smaller triangulation $G', f')$ with "the same" labeling L'. By minimality, C has a 1-uniform vertex x_1.

If $u_1 \notin V(C)$, then nothing changes at x_1. If $u_1 \in V(C)$, then C visits v_1 in G'. So, again x_1 is 1-uniform in C. $\square$

Theorem 4.18 (Tree Lemma)**.** *In every Schnyder labeling of a triangulation (G, f) for each $1 \le i \le 3$ the edges of color i form an $(n-2)$-vertex in-tree T_i with root v_i. Also, for each internal vertex v, the paths from v to v_i in T_i are internally disjoint for distinct i.*

Proof. Let T_i denote the subgraph G formed by the edges of color i. Then $|E(T_i)| = n-3$ and vertices v_{i-1} and v_{i+1} are not in T_i. Suppose first that T_i has a cycle $C = x_1x_2\dots x_kx_1$. Since only one edge of color i starts from each x_j, C is a directed cycle, say $\overrightarrow{x_jx_{j+1}} \in E(C)$ for each j. But then for each j label i is present at the end of each $\overrightarrow{x_jx_{j+1}}$ inside C, contradicting Theorem **??**.

Thus, T_i has $n-2$ vertices, $n-3$ edges and no cycles. So, it is a tree. Since no vertex apart from v_i is a sink in T_i, the tree is an in-tree with root v_i.

Suppose now that for $u \ne v$ there are v, u-paths in both T_1 and T_2. Choose such u and v so that the total length of the paths is minimum. Then these paths, say P_1 and P_2 form a cycle, say C. Note that at each internal vertex of P_1 there is an angle of color 1 and an angle of another color. The same for P_2 (with 1 switched to 2). Thus uniform vertices can be only u and v, but Theorem **??** says there are 3 such vertices, a contradiction. $\square$

—————————————— **Here Lecture 34 ended.**
—————————————— **Lecture 35 was presented by Mina Nahvi.**

Let $P_i(v)$ denote the v, v_i-path in T_i.

Let $R_i(v)$ denote the region enclosed by $P_{i-1}(v)$, $P_{i+1}(v)$ and edge $v_{i-1}v_{i+1}$.

Lemma 4.19. *Let $1 \le i \le 3$. If u and v are distinct internal vertices in a triangulation (G, f) and in a Schnyder labeling L of (G, f), $u \in R_i(v)$, then $R_i(u)$ is properly contained in $R_i(v)$.*

Proof. We may assume that $i = 1$. It is enough to show that neither of $P_2(u)$ and $P_3(u)$ has a vertex outside $R_1(v)$.

Consider first $P_2(u)$. If it comes to $P_2(v)$, then it must follow $P_2(v)$ till the very end. Suppose $P_2(u)$ hits $P_3(v) - P_2(v)$ at w. Then $w \neq v$, so there are edges $\overrightarrow{w'w}$ and $\overrightarrow{ww''}$ on $P_3(v)$ (of color 3). But then all edges of color 2 must hit w from the right, i.e. from $R_2(v)$, and the unique edge of color 2 leaving w goes to $R_1(v)$.

Symmetrically, if $P_3(u)$ comes to $P_3(v)$, then it then follows $P_3(v)$ till the very end, and every edge of color 3 leaving $P_2(v)$ goes to $R_1(v)$. $\square$

A digression.

A *barycentric representation* of G is an injection $\phi:\ V(G) \to \mathbf{R^3}$ s.t.

(i) the coordinates v_1, v_2, v_3 of $\phi(v)$ are nonnegative and $v_1+v_2+v_3 = 1$ for each $v \in V(G)$, and

(ii) if $uv \in E(G)$ and $w \in V(G) - u - v$, then $w_i > \max\{u_i, v_i\}$ for some $i \in [3]$.

Lemma 4.20. *If ϕ is a barycentric representation of G, then drawing the edges of G as straight segments connecting the images of the vertices yields a planar drawing of G.*

Proof. Recall that ϕ is an injection. Consider any two edges uv and wz with all 4 vertices distinct. By Part (ii) of the definition, there are indices i, j, h, k s.t.

$$u_i > \max\{w_i, z_i\},\ v_j > \max\{w_j, z_j\},\ w_h > \max\{u_h, v_h\},\ z_k > \max\{u_k, v_k\}.$$

By definition, $\{i, j\} \cap \{h, k\} = \emptyset$. By pigeonhole and symmetry, we may assume $i = j$. Then there is α s.t. both u and v are above the line $x_i = \alpha$, and both w and z are below it.

We also need to exclude the situation when say edge uv contains edge wv. If this would happen then (ii) would not hold. $\square$

Given a Schnyder labeling L of a triangulation (G, f), for each internal v let $r_i(v)$ denote the number of cells in $R_i(v)$. Also, for an external vertex v_i, let $r_i(v_i) = 2n-5$ and $r_j(v_i) = 0$ when $j \neq i$. Then $r_1(v) + r_2(v) + r_3(v) = 2n - 5$ for all $v \in V(G)$. Now define

$$\phi(v) = \left(\frac{r_1(v)}{2n-5}, \frac{r_2(v)}{2n-5}, \frac{r_3(v)}{2n-5}\right) \qquad \forall v \in V(G). \tag{2}$$

Theorem 4.21. *The function ϕ defined by* (**??**) *is a barycentric representation of G.*

Proof. Part (i) of the definition is clear. Suppose w is not an end of edge uv. If w is external, then (ii) is obvious. Suppose w is internal. Then u is in some $R_i(w)$ and v is in some $R_j(w)$. Since they are adjacent, either they both are in $R_i(w)$ or they both are in $R_j(w)$, say they both are in $R_i(w)$. Then $w_i > \max\{u_i, v_i\}$. $\square$

So, each planar graph has a straight-line embedding into the grid points of the triangle with corners $(0, 0)$, $(2n - 5, 0)$ and $(0, 2n - 5)$.

To shrink the size of the triangle, here is a refinement. We will closer follow the book notation for the homework.

—————————— **Here Lecture 36 ended.**

A *weak barycentric representation* of G is an injection $\phi:\ V(G) \to \mathbf{R^3}$ s.t.

(1) the coordinates v_1, v_2, v_3 of $\phi(v)$ are nonnegative and $v_1+v_2+v_3 = 1$ for each $v \in V(G)$, and

(2) if $xy \in E(G)$ and $z \in V(G) - x - y$, then for some $k \in [3]$ vectors (x_k, x_{k+1}) and (y_k, y_{k+1}) are lexicographically less than (z_k, z_{k+1}).

For an internal vertex v of a triangulation (G, f) with a Schnyder labeling L, let v'_i denote the number of vertices in $R_i(v) - P_{i-1}(v)$. Then $v'_1 + v'_2 + v'_3 = n - 1$. For an external vertex v_j, $R_j(v_j)$ has n vertices, $P_j(v_j)$ has one vertex and each of $P_{j-1}(v_j)$, $P_{j+1}(v_j)$ has two vertices. So, let $(v_j)'_j = n - 2$, $(v_j)'_{j+1} = 1$, and $(v_j)'_{j-1} = 0$.

TWO LEMMAS AND THEOREM IN HW5.

4.3. **Small separators in planar graphs.** .

An (m, α)-*separation* of G is a partition $V(G) = A \cup B \cup C$ s.t

(a) $|C| \leq m$,   (b) $G - C$ has no edges between A and B, and   (c) $|A|, |B| \leq \alpha |V(G)|$.

A class $\mathcal{F}$ of graphs is an f-separator with shrink factor α if each $G \in \mathcal{F}$ has an $(f(|V(G)|, \alpha)$-separation.

In general, for each $\epsilon > 0$ there is $c_\epsilon > 0$ s.t. for almost all G with $(2+\epsilon)k$ vertices and $c_\epsilon k$ edges deleting any k vertices results in a graph with a component with $\geq k$ vertices.

Lemma 4.22. *Let (G, f) be a near-triangulation with a 2-coloring of vertices with red and blue. If the outer cycle C has red vertices x, y, then G has either a red x, y-path or a blue path connecting the components of $C - x - y$.*

Proof. Consider the set A of the red vertices reachable from x via red paths. Consider its neighborhood. Use triangualtion (in class). □

Lemma 4.23. *Let (G, f) be a near-triangulation with the outer cycle $C = v_0, v_1, \ldots, v_{2k-1}, v_1$. If G has no v_0, v_k-path of length at most $k-1$, then there are $k-1$ disjoint paths $P_1, \ldots, P_{k-1}$ where P_i connects v_i with v_{2k-i}.*

Proof. Let S be a smallest set separating $X = \{v_1, \ldots, v_{k-1}\}$ from $Y = \{v_{k+1}, \ldots, v_{2k-1}\}$ in $G - v_0 - v_k$. Let **Red** $= S \cup \{v_0, v_k\}$. By definition, G has no blue X, Y-path. Then by Lemma **??**, G has a red v_0, v_k-path. But this path has at least $k - 1$ internal vertices, so $|S| \geq k - 1$. By Menger (Pym), there are $k - 1$ disjoint X, Y-paths. They form the linkage we promised, since (G, f) is plane. □

Theorem 4.24 (Lipton and Tarjan)**.** *For each $n \geq 1$ each n-vertex planar graph has a $(2\sqrt{2n}, 2/3)$-separation.*

Proof. So we prove the theorem for triangulations by induction on n. If $n < 30$, then simply delete any $\lfloor \sqrt{8n} \rfloor$ vertices. Let $k = \lfloor \sqrt{2n} \rfloor$.

—————————— **Here Lecture 37 ended.**

Define C^+ and C^-, $c^+ = |C^+|$ and $c^- = |C^-|$.

Among the cycles C with $c^+ \geq 2n/3$ and $|C| \leq 2k$, choose one with the minimum $c^- - c^+$.

If $c^- \le 2n/3$, we are done. Suppose $c^- > 2n/3$. Let $D = G[C^- \cup V(C)]$. For $u, v \in V(C)$, let $c(u,v)$ (resp., $d(u,v)$) be the distance between u and v in C (resp., in D). Since $C \subset D$, $c(u,v) \le d(u,v)$.

Claim 1: *$d(u,v) = c(u,v)$ for all $u, v \in V(C)$.*

Indeed, if not, choose a wrong pair (u,v) with minimum $d(u,v)$. Let P be a shortest u,v-path in G. By minimality, $V(P) \cap V(C) = \{u,v\}$. P forms with C two cycles, C_1 and C_2. Suppose $c_1^- \ge c_2^-$. By construction, $|C_1| \le |C| \le 2k$. Now,

$$n - c_1^+ = c_1^- + |V(C_1)| \ge \frac{c_1^- + c_2^-}{2} + 2|V(P)| - 1 > \frac{c^-}{2} + |V(P)| \ge \frac{n}{3}.$$

Thus, $c_1^+ \le \frac{2n}{3}$, contradicting the minimality of $c^- - c^+$.

Claim 2: *$|C| = 2k$.*

If shorter, we can make c^- smaller (using Claim 1).

So, let $C = v_0, v_1, \ldots, v_{2k-1}, v_1$. By Claim 1 and Lemma **??**, there are disjoint paths $P_1, \ldots, P_{k-1}$, where P_i is a v_i, v_{2k-i}-path. Again by Claim 1, $|V(P_i)| \ge 1 + \min\{2i, 2(k-i)\}$. Hence

$$|V(G)| > |V(D)| \ge (1+3+\ldots+\lceil((k+1)/2)^2\rceil) + (1+3+\ldots+\lfloor((k+1)/2)^2\rfloor) \ge \frac{(k+1)^2}{2} > n,$$

a contradiction. □

Applications.

4.4. **Discharging for planar graphs.** .

Examples of discharging, versions of Euler's Formula. The FCT.

A *normal plane map* is a connected plane multigraph whose vertex degrees and face lengths all are at least 3.

Lebesgue (1940) proved that each 3-connected plane graph (G,f) with $\delta(G) \ge 5$ has a 3-face (a,b,c) with $d(a) + d(b) + d(c) \le 19$. Kotzig (1963) improved 19 to 18 and in 1979 conjectured 17. An example of 17 is obtained from the dodecahedron by inserting a vertex into each face.

Theorem 4.25 (Borodin, 1989)**.** *Every normal plane map (G,f) with $\delta(G) \ge 5$ has a 3-face (a,b,c) with $d(a) + d(b) + d(c) \le 17$.*

———————————— **Here Lecture 38 ended.**

Proof. For a given n, consider an edge maximal counter-example (G,f).

Claim: *For each 4^+-face F, $d(v) = 5$ for every $v \in F$ (by the maximality of (G,f)).*

(Here normal maps are used.)

Define the initial charge: $ch(v) = d(v) - 6$ for each $v \in V(G)$ and $ch(\alpha) = 2d(\alpha) - 6$ for each face of (G,f).

Discharging:

(R1) Each 4^+-face gives $1/2$ to each its vertex.

(R2) Each 7-vertex gives $1/3$ to each its 5-neighbor.

(R3) Each 8^+-vertex gives $1/4$ to each incident 3-face and then each such 3-face shares the obtained surplus among its 5-vertices.

We will prove that the new charge ch^* is nonnegative for each vertex and face.

First, look at faces. Each 3-face has initial charge 0 and gives out only a surplus. For $k \geq 4$ any k-face α gives out by (R1) exactly $k/2$ and remains with the charge $(2k-6) - k/2 = \frac{3}{2}(k-4) \geq 0$.

Now, look at vertices. For $k \geq 8$ any k-vertex v gives out by (R3) at most $k/4$ and remains with the charge at least $k - 6 - k/4 = \frac{3}{4}(k-8) \geq 0$.

Suppose $d(v) = 7$. By the claim, all faces containing v are triangles. Hence no two consecutive neighbors are 5-vertices. It follows that has at most 3 5-neighbors. Thus, $ch^*(v) \geq 7 - 6 - 3(1/3) = 0$.

If $d(v) = 6$, then v keeps its original charge of 0.

Finally, suppose $d(v) = 5$ and the neighbors of v are $x_1, \ldots, x_5$ in clockwise order. At the start, its charge is -1. If v is incident with at least two 4^+-faces, then by (R1) it will get from them $2(1/2) = 1$. Let now v be incident with exactly one 4^+-face α and let x_1, v, x_5 be a part of the boundary of α (see the picture in class). By the claim, $d(x_1) = d(x_5) = 5$. Then $d(x_2) \geq 8$ and $d(x_4) \geq 8$, so v will get from them by (R3) at least $4((1/4)/2) = 1/2$ and get $1/2$ from α.

Now, assume all faces incident to v are 3-faces. Then at most two neighbors of v are 5-vertices. If exactly two, then the remaining neighbors are 8^+-vertices that together will give 1 to v (see pictures in class). Suppose only x_1 is the 5-neighbor of v. Then x_2 and x_5 are 8^+-vertices and one of x_3, x_4 is a 7^+-neighbor, say $d(x_3) \geq 7$. Then x_5 gives to v $1/8$ via face vx_5x_1 and $1/4$ via face vx_5x_4. Similarly, x_1 gives to v $3/8$. Vertex x_3 gives to v $1/3$ if $d(x_3) = 7$ and $1/2$ if $d(x_3) \geq 8$.

The last subcase of the last case is that v has no 5-neighbors. Then it has at least 3 7^+-neighbors, and each of them gives to v at least $1/3$. □

Recall:

Theorem 4.26 (Grötzsch)**.** *Every planar graph with no triangles is* 3*-colorable.*

Conjecture (Steinberg, 1976). *Every planar graph with no* 4*-cycles and* 5*-cycles is* 3*-colorable.*

Question (Erdős, 1993). *Does there exists $k \geq 5$ such that every planar graph with no cycles of length $4, 5, \ldots, k$ is* 3*-colorable?*

Abbott and Zhou: $k = 11$ works.

———————————— **Here Lecture 39 ended.**

Borodin (1996), Sanders and Zhao (1995): $k = 9$ works.

Borodin, Glebov, Raspaud and Salavatipour (2006): $k = 7$ works.

Voigt (2005): Not for list coloring.

Cohen, Addad, Hebdige, Král, Li and Salgado (2017): $k = 5$ DOES NOT WORK.

We will prove that $k = 9$ works.

Lemma 4.27 (Borodin (1996))**.** *Every plane graph (G, f) with $\delta(G) \geq 3$ has*

(i) two 3*-faces with a common edge, or*

(ii) a j-face for some $4 \le j \le 9$, or
(iii) a 10-face with all vertices of degree 3.

Proof. Suppose none of (a)–(c) holds. Apply the balanced Euler's Formula:

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{\alpha \in F(G,f)} (d(\alpha) - 4) = -8.$$

For each $x \in V(G) \cup F(G, f)$, the initial charge is $ch(x) = d(x) - 4$. The discharging rules are:
(R1) Each 3-face gets $1/3$ from each incident vertex.
(R2) Each non-3-face α ($d(\alpha) \ge 10$) gives:
(a) $2/3$ to each incident 3-vertex incident to a 3-face,
(b) $1/3$ to each other incident 3-vertex,
(c) $1/3$ to each incident 4-vertex on two 3-faces,
(d) $1/3$ to each incident 4-vertex that has a 3-face opposite to α.

Let us check that the new charge $ch^*(x)$ is nonnegative for each $x \in V(G) \cup F(G, f)$ (which would be a contradiction). Consider all cases for x:
(A) x is a 3-face. By (R1), $ch^*(x) = (3 - 4) + 3(1/3) = 0$.
(B) x is a 3-vertex. If x is in no 3-faces, then by (R2)(b), $ch^*(x) = (3 - 4) + 3(1/3) = 0$. If there is an incident 3-face (then only one!), then x gives to it $1/3$ by (R1), but gets $2(2/3) = 4/3$ by (R2)(a).
(C) x is a 4-vertex. If x is in no 3-faces, then $ch^*(x) = ch(x) = 0$. If there is exactly one incident 3-face, then x gives to it $1/3$ by (R1), but gets $1/3$ by (R2)(d). If there are two incident 3-faces (it cannot be more), then x gives to them $2(1/3)$ by (R1), but gets $2(1/3)$ by (R2)(c).
(D) x is a j-vertex for some $j \ge 5$. Then x belongs to at most $\lfloor j/2 \rfloor$ 3-faces, so by (R1), $ch^*(x) \ge (j - 4) - \lfloor j/2 \rfloor (1/3) > 0$.
(E) x is a j-face for some $j \ge 10$, say $x = v_i, v_2, \dots, v_j, v_1$. If x gives $2/3$ to some v_i, then $d(v_i) = 3$ and v_i belongs to a 3-face α.

(3) *This α shares exactly one edge with x.*

Thus when j is odd, x cannot give $2/3$ to each its vertex. Hence $ch^*(x) \ge (j - 4) - j(2/3)$ when j is even and $ch^*(x) \ge (j - 4) - j(2/3) + 1/3$ when j is odd. This is nonnegative for all $j \ge 11$.

Let $j = 10$ and exactly i vertices of x be 3-vertices on 3-faces. If $i = 10$, then Part (iii) of the lemma holds. If $i \le 8$, then $ch^*(x) \ge (10-4) - 8(2/3) - 2(1/3) = 0$. By (**??**), $i \ne 9$. □

Theorem 4.28 (Borodin, Sanders and Zhao). *Every planar graph with no cycles of length $4, 5, \dots, k$ is 3-colorable.*

Proof. If not, then there is a counter-example (G, f) with the fewest vertices. By minimality, G is 4-critical, so it is 2-connected and $\delta(G) \ge 3$. By Lemma **??**, G has a 10-cycle C with $|C| = 10$. Since G is 4-critical, $G - V(C)$ has a 3-coloring ϕ. Extend it to C (in class). □