4. Lecture notes: Planar graphs

4.1. Basics, classic theorems. A polygonal curve is a curve composed of finitely many line segments.

A drawing of a graph $G$ is a function $f : V(G) \cup E(G) \to \mathbb{R}^2$ s.t.
(a) $f(v) \in \mathbb{R}^2$ for every $v \in V(G)$;
(b) $f(v) \neq f(v')$ if $v, v' \in V(G)$ and $v \neq v'$;
(c) $f(xy)$ is a polygonal curve connecting $f(x)$ with $f(y)$.

A crossing in a drawing of a graph is a common vertex in the images of two edges that is not the image of their common end.

A graph $G$ is planar if it has a drawing $f$ without crossings.

A plane graph is a pair $(G, f)$ where $f$ is a drawing of $G$ without crossings.

A face of a plane graph $(G, f)$ is a connected component of $\mathbb{R}^2 - f(V(G) \cup E(G))$.

The length, $\ell(F_i)$, of a face $F_i$ in a plane graph $(G, f)$ is the total length of the closed walk(s) bounding $F_i$.

**Restricted Jordan Curve Theorem:** A simple closed polygonal curve $C$ in the plane partitions the plane into exactly two faces each having $C$ as boundary.

By $F(G, f)$ we denote the set of faces of the plane graph $(G, f)$.

**Proposition 4.1.** For each plane graph $(G, f)$,

\[
\sum_{F_i \in F(G, f)} \ell(F_i) = 2|E(G)|.
\]

**Proof.** By the definition of $\ell(F_i)$, each edge either contributes 1 to the length of two distinct faces or contributes 2 to the length of one face.

**Theorem 4.2** (Euler’s Formula). For every connected plane graph $(G, f)$,

\[
|V(G)| - |E(G)| + |F(G, f)| = 2.
\]

**Corollary 4.3.** For $n \geq 3$, every simple planar $n$-vertex graph $G$ has at most $3n - 6$ edges. Moreover, if $G$ is triangle-free, then $G$ has at most $2n - 4$ edges.

Corollary 4.4. Graphs $K_5$ and $K_{3,3}$ are not planar.

A Kuratowski graph is a subdivision of $K_5$ or $K_{3,3}$. It follows from Euler’s Formula that neither $K_5$ nor $K_{3,3}$ is planar. Thus every Kuratowski graph is nonplanar. Our goal is to prove the following classic theorem.

**Theorem 4.5** (Kuratowski, 1930). A graph $G$ is planar if and only if $G$ does not contain a Kuratowski subgraph.

The “only if” part is already proved. Let us prove the “if” part.

**Claim 4.6.** For every graph $G$ and any $xy \in E(G)$, if $G$ does not contain a Kuratowski subgraph, then $G/xy$ also doesn’t.
Proof. Suppose that $G/xy$ contains a Kuratowski subgraph $H$. Let $z$ be the vertex resulting from contracting $x$ with $y$. If $z \notin V(H)$, then $H$ is a Kuratowski subgraph of $G$. If $z \in V(H)$ but is not a branch vertex of $H$, then we can obtain a Kuratowski subgraph $H'$ of $G$ by replacing $z$ in $H$ with either $x$, or $y$, or \{x, y\}. The same holds if $z$ is a branch vertex of $H$, and at most one edge of $H$ incident with $z$ is incident with $x$ in $G$. Thus the remaining case is that $H$ is a subdivision of $K_5$ and exactly two edges of $H$ incident with $z$ are incident with $x$ in $G$ (see Fig. 1 (left)).

![Diagram](image-url)

**Figure 1**

Then $G$ contains a subdivision of $K_{3,3}$ as in Fig. 1 (right). □

First, we will prove a stronger statement for 3-connected graphs. A convex embedding of a planar graph $G$ is one in which every edge of $G$ forms a straight segment and every face (including the outer face) is a convex polygon. Not every planar graph has a convex embedding; for example, $K_{2,4}$ has not.

**Theorem 4.7** (Tutte). Every 3-connected graph with no Kuratowski subgraph has a convex embedding in the plane with no three vertices on a line.

**Proof.** By induction on $n := |V(G)|$. If $n \leq 4$, then the only 3-connected graph is $K_4$, and $K_4$ has such embedding.

Suppose the theorem holds for all graphs with at most $n - 1$ vertices. Let $G$ be any $n$-vertex 3-connected graph with no Kuratowski subgraph. By Contraction Lemma (7.2.7 in the book), $G$ has an edge $xy$ such that $H := G/xy$ is 3-connected. By Claim 4.6, $H$ has no Kuratowski subgraph. So by the IH, $H$ has a convex embedding in the plane with no three vertices on a line. Fix such an embedding. Let $z$ be the result of contracting $xy$ and $H'$ be obtained from $H$ by deleting all edges incident with $z$. Since $H' - z$ is 2-connected, the face $C$ of $H'$ containing $z$ is a cycle. Let $x_1, \ldots, x_k$ be the neighbors of $x$ on $C$ in cyclic order. If there is some $i$ such that all neighbors of $y$ on $C$ are in the portion of $C$ between $x_i$ and $x_{i+1}$, then we can obtain a convex embedding of $G$ with no three vertices on a line by placing $x$ into the position of $z$ and placing $y$ very close to $x$. If this does not happen, then either (a) $y$ and $x$ have 3 common neighbors, say $u, v, w$, or (b) for some $i < j$, $y$ has a neighbor $v$ on $C$ between $x_i$ and $x_j$ (in clockwise order) and a neighbor $u$ between $x_j$ and $x_i$.

In Case (a) we have a $K_5$-subdivision and in Case (b) we have a $K_{3,3}$-subdivision. □
In order to prove Theorem 4.5, it is now enough to show the following.

**Lemma 4.8.** If $G$ has the fewest vertices among the nonplanar graphs with no Kuratowski subgraphs, then $G$ is 3-connected.

**Proof.** We need the following simple observation:

(**) If $F$ is a face in an embedding of a graph $G$ in the plane, then there is an embedding of $G$ in the plane where $F$ the outer face.

If $G$ is disconnected, then by the minimality of $G$, each of its components could be embedded in the plane. The union of these embeddings will be an embedding of $G$. Suppose $G$ has a cut vertex $x$ and $H$ is a component of $G - x$. Let $H_1 = G[V(H) + x]$ and $H_2 = G - H$. By the minimality of $G$, each of $H_1$ and $H_2$ could be embedded in the plane. Then by (**), each of $H_1$ and $H_2$ has an embedding in the plane such that $x$ is the outer face. Stretching each of these embeddings so that each of the graphs is in one half-plane passing through $x$, we can then glue them into an embedding of $G$.

Suppose now that $G$ is 2-connected and that sets $V_1, V_2 \subset V(G)$ and vertices $x, y$ are such that $V_1 \cup V_2 = V(G)$, $V_1 \cap V_2 = \{x, y\}$ and there are no edges between $V_1 - x - y$ and $V_2 - x - y$. For $i = 1, 2$, let $G_i$ be the graph obtained from $G[V_i]$ by adding edge $xy$. If both $G_1$ and $G_2$ are planar, then by (**), there are their embeddings with edge $xy$ on the outer face. Again, we can stretch these embeddings so that we can glue them along $xy$ and get an embedding of $G$. Thus we may assume that $G_1$ is not planar. By the minimality of $G$, $G_1$ contains a Kuratowski subgraph $H$. Since $G$ does not contain Kuratowski subgraphs, $H$ contains edge $xy$. So we can get a Kuratowski subgraph $H'$ of $G$ from $H$ by replacing $xy$ with an $x, y$-path in $G[V_2]$. Such an $x, y$-path exists, since $G$ is 2-connected and so each of $x$ and $y$ has a neighbor in every component of $G - x - y$. □

\[\text{Here Lecture 31 ended.}\]

**Theorem 4.9** (Wagner, 1937). A graph $G$ is planar if and only if $G$ does not contain a subgraph contractible to $K_5$ or $K_{3,3}$.

**Proof.** The difficult part by Kuratowski’s Theorem. □

The cycle space, $\mathcal{C}(G)$, of a graph $G$ is the set of characteristic vectors of even subgraphs, i.e. of edge-disjoint unions of cycles in $G$.

The bond space, $\mathcal{B}(G)$, of a graph $G$ is the set of characteristic vectors of edge cuts in $G$.

**Theorem 4.10.** The cycle space and the bond space of a connected $n$-vertex graph $G$ with $m$ edges are binary vector spaces with dimensions $m - n + 1$ and $n - 1$, resp. They are orthogonal complements to each other in $\mathbb{R}^m$.

**Proof.** Check that the sum of char. vectors of even subgraphs (resp. of edge cuts) is again a char. vector of an even subgraph (resp. of an edge cut).

Fix a spanning tree $T$. Each $e \in E(G) - E(T)$ forms a cycle with a part of $T$, and the char. vectors of all these cycles are linearly independent. Thus, $\dim(\mathcal{C}(G)) \geq m - n + 1$.

Fix a vertex $v \in V(G)$. For each $w \in V(G) - v$, let $B_w$ be the edge cut separating $w$ from the rest. The char. vectors of all these cuts are linearly independent. Thus, $\dim(\mathcal{B}(G)) \geq n - 1$. 

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Since each cycle intersects each edge cut in an even number of edges, \( C \perp B \) for any \( C \subseteq C \) and \( B \subseteq B \). So by Rank-Nullity Theorem, \( \dim(C(G)) + \dim(B(G)) \leq m \). Thus, \( \dim(C(G)) = m - n + 1 \) and \( \dim(B(G)) = n - 1 \). \( \square \)

A 2-basis for a linear subspace \( L \) of a space with a given basis \( B \) is a basis of \( L \) s.t. each coordinate is non-zero in at most two vectors of this basis.

**Theorem 4.11** (MacLane). A graph \( G \) is planar iff \( C(G) \) has a 2-basis.

**Proof.** \( (\Rightarrow) \) If \( G \) is not 2-connected, then any basis of \( C(G) \) is the disjoint union of bases of cycle spaces of its blocks. If \( G \) is planar and 2-connected, and \( f \) is its planar drawing, then the facial cycles of all its bounded faces form a 2-basis for \( C(G) \).

\( (\Leftarrow) \) Suppose \( G \) is not planar. Then it has a subdivision of \( K_5 \) or \( K_{3,3} \).

**Claim 1:** \( C(K_5) \) has no 2-basis.

**Proof of Claim 1:** By Theorem 4.10, a basis of \( C(K_5) \) contains \( m - n + 1 = 10 - 5 + 1 = 6 \) even graphs \( C_1, \ldots, C_6 \). Let \( C_0 = \sum_{i=1}^{6} C_i \). Note that each edge of \( G \) is in at most two of \( C_0, \ldots, C_6 \). Also \( C_0 \neq \emptyset \), since it is a nontrivial sum of basis vectors. But
\[
\sum_{i=0}^{6} |C_i| \geq 7 \cdot 3 = 21 > 2|E(K_5)|.
\]

**Claim 2:** \( C(K_{3,3}) \) has no 2-basis.

**Proof of Claim 2:** Repeat the proof of Claim 1, but the length of each cycle is now at least 4.

**Claim 3:** The space \( C(H) \) of a subdivision \( H \) of a graph \( G \) has a 2-basis iff the space \( C(G) \) has a 2-basis.

**Proof of Claim 3:** Check for subdividing an edge.

**Claim 4:** If \( C(G) \) has a 2-basis, then for any \( e \in E(G) \), \( C(G - e) \) has a 2-basis.

**Proof of Claim 4:** If \( e \) is a cut edge, then \( C(G - e) = C(G) \). Suppose \( e \) is not. Then \( \dim(C(G - e)) = \dim(C(G)) - 1 \). Let \( C = \{C_1, \ldots, C_k\} \) be a 2-basis of \( C(G) \).

If \( e \in C_1 \) and to no other \( C_i \), then \( C' = C - C_1 \) is a 2-basis of \( C(G - e) \). If \( e \in C_1 \cap C_2 \), then \( C' = C - \{C_1, C_2\} \cup (C_1 + C_2) \) is a 2-basis of \( C(G - e) \). This proves Claim 4.

The claims together with the fact that \( G \) has a subdivision of \( K_5 \) or \( K_{3,3} \) prove the theorem. \( \square \)

A bond is an edge cut whose edge set does not contain edge sets of other nontrivial edge cuts.

For a multigraph \( G \), a multigraph \( H \) is an *abstract dual* to \( G \) if there is a bijection \( f : E(G) \rightarrow E(H) \) s.t.
\( X \subseteq E(G) \) is a cycle in \( G \) \( \iff \) \( f(X) \) is a bond in \( H \).

**Theorem 4.12.** A graph \( G \) is planar iff \( G \) has an abstract dual.

**Proof.** First, observe that \( G \) is planar iff each its block is planar. Also, \( G \) has an abstract dual iff each its block has an abstract dual and the images of edge sets of distinct blocks are disjoint. So, we prove the theorem for 2-connected graphs.

\( (\Rightarrow) \) If \( G \) is planar, then its geometric dual is its abstract dual.
Suppose $G$ has an abstract dual $H$ (using map $f$). The basis for $\mathcal{B}(H)$ constructed in the proof of Theorem 4.10 is a 2-basis. Since $f$ creates a bijection between $\mathcal{C}(G)$ and $\mathcal{B}(H)$, $\mathcal{C}(G)$ has a 2-basis. □

4.2. Schnyder labelings. Let $(G, f)$ be a triangulation. Then a cell is a bounded face. An angle in a cell $c$ is a pair $(c, v)$ where $v$ is a vertex of this cell, so each cell has 3 angles, and each vertex $v$ is in $d(v)$ angles.

A Schnyder labeling of a triangulation $(G, f)$ is a labeling of the angles in each cell with 1, 2 and 3 such that

1. the angles in each cell are labeled with 1, 2 and 3 in clockwise order, and
2. each interior vertex has angles with each label appearing in clockwise order: first all ones, then all two’s and the all 3’s.

Example!

Observation: Given a Schnyder labeling of a triangulation $(G, f)$, if two cells abc and abd share edge ab and their clockwise orders are $a, c, b$ and $a, b, d$, then the labels at $a$ are distinct, and the labels at $b$ coincide, and differ from the labels at $a$.

This allows us to define an orientation and an edge coloring of $(G, f)$ (see the book). In particular, the outdegree of each internal vertex is 3.

Lemma 4.13. The external vertices can be labeled $v_1, v_2, v_3$ so that for each $1 \leq i \leq 3$ all internal angles at $v_i$ have label $i$.

Proof. Since $G$ has $3n - 9$ internal edges and from each of the $n - 3$ internal vertices start 3 edges, the directed edges only enter the exterior vertices. □

Call an internal edge of a triangulation contractible if its end vertices have only two common neighbors.

Lemma 4.14. If $a$ is an external vertex of a triangulation $(G, f)$ with $|V(G)| \geq 4$, then some internal edge $au$ is contractible.

Proof. Let the neighborhood of $a$ is a cycle $C = x_1 x_2 \ldots x_k x_1$ where $x_1, x_k$ are external (draw a picture!). Choose a shortest chord $x_ix_{i+1}$ of the path $C - x_1 x_k$. Then $ax_{i+1}$ is contractible. □

Here Lecture 33 ended.

Theorem 4.15. Each triangulation has a Schnyder labeling.

Proof. By induction with contractions (see the book). □

The next lemma shows that all Schnyder labelings appear “this way”.

Lemma 4.16. Let $L$ be a Schnyder labeling of a triangulation $(G, f)$ with $|V(G)| \geq 4$. Then for each $1 \leq i \leq 3$, $v_i$ has an internal neighbor $u_i$ s.t.

(a) $v_i u_i$ is contractible and
(b) all internal angles at $u_i$ not involving $v_i$ are labeled by $i$. 

Theorem 4.17 (Uniform Angle Lemma). In every Schnyder labeling of a triangulation $(G, f)$ for each $1 \leq i \leq 3$ and each cycle $C$ in $G$, there is an $i$-uniform vertex $x_i$ on $C_i$, i.e. all angles at $x_i$ inside $C$ have label $i$.

Proof. By induction on $n = |V(G)|$. If $C$ visits all external vertices, then O.K. (In particular, $n > 3$.) Otherwise, suppose $v_1 \notin V(C)$. By Lemma 4.16, there is $u_1 \in N(v_1) - v_2 - v_3$ s.t. contracting $v_1u_1$ leads to a smaller triangulation $G', f'$ with “the same” labeling $L'$. By minimality, $C$ has a 1-uniform vertex $x_1$.

If $u_1 \notin V(C)$, then nothing changes at $x_1$. If $u_1 \in V(C)$, then $C$ visits $v_1$ in $G'$. So, again $x_1$ is 1-uniform in $C$. □

Theorem 4.18 (Tree Lemma). In every Schnyder labeling of a triangulation $(G, f)$ for each $1 \leq i \leq 3$ the edges of color $i$ form an $(n - 2)$-vertex in-tree $T_i$ with root $v_i$. Also, for each internal vertex $v$, the paths from $v$ to $v_i$ in $T_i$ are internally disjoint for distinct $i$.

Proof. Let $T_i$ denote the subgraph $G$ formed by the edges of color $i$. Then $|E(T_i)| = n - 3$ and vertices $v_{i-1}$ and $v_{i+1}$ are not in $T_i$. Suppose first that $T_i$ has a cycle $C = x_1x_2 \ldots x_kx_1$. Since only one edge of color $i$ starts from each $x_j$, $C$ is a directed cycle, say $x_jx_{j+1} \in E(C)$ for each $j$. But then for each $j$ label $i$ is present at the end of each $x_jx_{j+1}$ inside $C$, contradicting Theorem 4.17.

Thus, $T_i$ has $n - 2$ vertices, $n - 3$ edges and no cycles. So, it is a tree. Since no vertex apart from $v_i$ is a sink in $T_i$, the tree is an in-tree with root $v_i$.

Suppose now that for $u \neq v$ there are $v, u$-paths in both $T_1$ and $T_2$. Choose such $u$ and $v$ so that the total length of the paths is minimum. Then these paths, say $P_1$ and $P_2$ form a cycle, say $C$. Note that at each internal vertex of $P_1$ there is an angle of color 1 and an angle of another color. The same for $P_2$ (with 1 switched to 2). Thus uniform vertices can be only $u$ and $v$, but Theorem 4.17 says there are 3 such vertices, a contradiction. □

Here Lecture 34 ended.

Lecture 35 was presented by Mina Nahvi.

Let $P_i(v)$ denote the $v, v_i$-path in $T_i$.
Let $R_i(v)$ denote the region enclosed by $P_{i-1}(v), P_{i+1}(v)$ and edge $v_{i-1}v_{i+1}$.

Lemma 4.19. Let $1 \leq i \leq 3$. If $u$ and $v$ are distinct internal vertices in a triangulation $(G, f)$ and in a Schnyder labeling $L$ of $(G, f)$, $u \in R_i(v)$, then $R_i(u)$ is properly contained in $R_i(v)$.
Proof. We may assume that \( i = 1 \). It is enough to show that neither of \( P_2(u) \) and \( P_3(u) \) has a vertex outside \( R_1(v) \).

Consider first \( P_2(u) \). If it comes to \( P_2(v) \), then it must follow \( P_2(v) \) till the very end. Suppose \( P_3(u) \) hits \( P_3(v) - P_2(v) \) at \( w \). Then \( w \neq v \), so there are edges \( \overrightarrow{ww} \) and \( \overrightarrow{ww}'' \) on \( P_3(v) \) (of color 3). But then all edges of color 2 must hit \( w \) from the right, i.e. from \( R_2(v) \), and the unique edge of color 2 leaving \( w \) goes to \( R_1(v) \).

Symmetrically, if \( P_3(u) \) comes to \( P_3(v) \), then it then follows \( P_3(v) \) till the very end, and every edge of color 3 leaving \( P_2(v) \) goes to \( R_1(v) \). □

A digression.

A barycentric representation of \( G \) is an injection \( \phi : V(G) \to \mathbb{R}^3 \) s.t.

(i) the coordinates \( v_1, v_2, v_3 \) of \( \phi(v) \) are nonnegative and \( v_1 + v_2 + v_3 = 1 \) for each \( v \in V(G) \), and

(ii) if \( uv \in E(G) \) and \( w \in V(G) - u - v \), then \( w_i > \max\{u_i, v_i\} \) for some \( i \in [3] \).

Lemma 4.20. If \( \phi \) is a barycentric representation of \( G \), then drawing the edges of \( G \) as straight segments connecting the images of the vertices yields a planar drawing of \( G \).

Proof. Recall that \( \phi \) is an injection. Consider any two edges \( uv \) and \( wz \) with all 4 vertices distinct. By Part (ii) of the definition, there are indices \( i, j, h, k \) s.t.

\[
\begin{align*}
  u_i &> \max\{w_i, z_i\}, \quad v_j > \max\{w_j, z_j\}, \quad w_h > \max\{u_h, v_h\}, \quad z_k > \max\{u_k, v_k\}.
\end{align*}
\]

By definition, \( \{i, j\} \cap \{h, k\} = \emptyset \). By pigeonhole and symmetry, we may assume \( i = j \). Then there is \( \alpha \) s.t. both \( u \) and \( v \) are above the line \( x_i = \alpha \), and both \( w \) and \( z \) are below it.

We also need to exclude the situation when say edge \( uv \) contains edge \( wz \). If this would happen then (ii) would not hold. □

Given a Schnyder labeling \( L \) of a triangulation \( (G, f) \), for each internal \( v \) let \( r_i(v) \) denote the number of cells in \( R_i(v) \). Also, for an external vertex \( v_i \), let \( r_i(v_i) = 2n - 5 \) and \( r_j(v_i) = 0 \) when \( j \neq i \). Then \( r_1(v) + r_2(v) + r_3(v) = 2n - 5 \) for all \( v \in V(G) \). Now define

\[
\phi(v) = \left( \frac{r_1(v)}{2n - 5}, \frac{r_2(v)}{2n - 5}, \frac{r_3(v)}{2n - 5} \right) \quad \forall v \in V(G).
\]

Theorem 4.21. The function \( \phi \) defined by (2) is a barycentric representation of \( G \).

Proof. Part (i) of the definition is clear. Suppose \( w \) is not an end of edge \( uv \). If \( w \) is external, then (ii) is obvious. Suppose \( w \) is internal. Then \( u \) is in some \( R_i(w) \) and \( v \) is in some \( R_j(w) \). Since they are adjacent, either they both are in \( R_i(w) \) or they both are in \( R_j(w) \), say they both are in \( R_i(w) \). Then \( w_i > \max\{u_i, v_i\} \). □

So, each planar graph has a straight-line embedding into the grid points of the triangle with corners \((0, 0)\), \((2n - 5, 0)\) and \((0, 2n - 5)\).

To shrink the size of the triangle, here is a refinement. We will closer follow the book notation for the homework.
Here Lecture 36 ended.

A weak barycentric representation of $G$ is an injection $\phi : V(G) \to \mathbb{R}^3$ s.t.
(1) the coordinates $v_1, v_2, v_3$ of $\phi(v)$ are nonnegative and $v_1 + v_2 + v_3 = 1$ for each $v \in V(G)$, and
(2) if $xy \in E(G)$ and $z \in V(G) - x - y$, then for some $k \in [3]$ vectors $(x_k, x_{k+1})$ and $(y_k, y_{k+1})$ are lexicographically less than $(z_k, z_{k+1})$.

For an internal vertex $v$ of a triangulation $(G, f)$ with a Schnyder labeling $L$, let $v_i'$ denote the number of vertices in $R_i(v) - P_{i-1}(v)$. Then $v_1' + v_2' + v_3' = n - 1$. For an external vertex $v_j$, $R_j(v_j)$ has $n$ vertices, $P_j(v_j)$ has one vertex and each of $P_{i-1}(v_j), P_{i+1}(v_j)$ has two vertices. So, let $(v_j)_j' = n - 2$, $(v_j)'_{j+1} = 1$, and $(v_j)'_{j-1} = 0.

**TWO LEMMAS AND THEOREM IN HW5.**

4.3. Small separators in planar graphs.

An $(m, \alpha)$-separation of $G$ is a partition $V(G) = A \cup B \cup C$ s.t

(a) $|C| \leq m$,
(b) $G - C$ has no edges between $A$ and $B$, and
(c) $|A|, |B| \leq \alpha|V(G)|$.

A class $F$ of graphs is an $f$-separator with shrink factor $\alpha$ if each $G \in F$ has an $(f(|V(G)|, \alpha)$-separation.

In general, for each $\epsilon > 0$ there is $c_\epsilon > 0$ s.t. for almost all $G$ with $(2 + \epsilon)k$ vertices and $c_\epsilon k$ edges deleting any $k$ vertices results in a graph with a component with $\geq k$ vertices.

**Lemma 4.22.** Let $(G, f)$ be a near-triangulation with a 2-coloring of vertices with red and blue. If the outer cycle $C$ has red vertices $x, y$, then $G$ has either a red $x, y$-path or a blue path connecting the components of $C - x - y$.

**Proof.** Consider the set $A$ of the red vertices reachable from $x$ via red paths. Consider its neighborhood. Use triangualtion (in class). □

**Lemma 4.23.** Let $(G, f)$ be a near-triangulation with the outer cycle $C = v_0, v_1 \ldots, v_{2k-1}, v_1$. If $G$ has no $v_0, v_k$-path of length at most $k - 1$, then there are $k - 1$ disjoint paths $P_1, \ldots, P_{k-1}$ where $P_i$ connects $v_i$ with $v_{2k-i}$.

**Proof.** Let $S$ be a smallest set separating $X = \{v_1, \ldots, v_{k-1}\}$ from $Y = \{v_{k+1}, \ldots, v_{2k-1}\}$ in $G - v_0 - v_k$. Let $\text{Red} = S \cup \{v_0, v_k\}$. By definition, $G$ has no blue $X, Y$-path. Then by Lemma 4.22, $G$ has a red $v_0, v_k$-path. But this path has at least $k - 1$ internal vertices, so $|S| \geq k - 1$. By Menger (Pym), there are $k - 1$ disjoint $X, Y$-paths. They form the linkage we promised, since $(G, f)$ is plane. □

**Theorem 4.24** (Lipton and Tarjan). For each $n \geq 1$ each $n$-vertex planar graph has a $(2\sqrt{2n}, 2/3)$-separation.

**Proof.** So we prove the theorem for triangulations by induction on $n$. If $n < 30$, then simply delete any $\lceil \sqrt{8n} \rceil$ vertices. Let $k = \lceil \sqrt{2n} \rceil$.

Here Lecture 37 ended.

Define $C^+$ and $C^-$, $c^+ = |C^+|$ and $c^- = |C^-|$.

Among the cycles $C$ with $c^+ \geq 2n/3$ and $|C| \leq 2k$, choose one with the minimum $c^- - c^+$. 8
If \( c^- \leq 2n/3 \), we are done. Suppose \( c^- > 2n/3 \). Let \( D = G[C^- \cup V(C)] \). For \( u, v \in V(C) \), let \( c(u, v) \) (resp., \( d(u, v) \)) be the distance between \( u \) and \( v \) in \( C \) (resp., in \( D \)). Since \( C \subset D \), \( c(u, v) \leq d(u, v) \).

**Claim 1:** \( d(u, v) = c(u, v) \) for all \( u, v \in V(C) \).

Indeed, if not, choose a wrong pair \((u, v)\) with minimum \( d(u, v) \). Let \( P \) be a shortest \( u, v \)-path in \( G \). By minimality, \( V(P) \cap V(C) = \{u, v\} \). \( P \) forms with \( C \) two cycles, \( C_1 \) and \( C_2 \). Suppose \( c^-_1 \geq c^-_2 \). By construction, \( |C_1| \leq |C| \leq 2k \). Now,

\[
n - c^+_1 = c^-_1 + |V(C_1)| \geq \frac{c^-_1 + c^-_2}{2} + 2|V(P)| - 1 > \frac{c^-}{2} + |V(P)| \geq \frac{n}{3}.
\]

Thus, \( c^+_1 \leq \frac{2n}{3} \), contradicting the minimality of \( c^- - c^+ \).

**Claim 2:** \( |C| = 2k \).

If shorter, we can make \( c^- \) smaller (using Claim 1).

So, let \( C = v_0, v_1, \ldots, v_{2k-1}, v_1 \). By Claim 1 and Lemma 4.23, there are disjoint paths \( P_1, \ldots, P_{k-1} \), where \( P_i \) is a \( v_i, v_{2k-1} \)-path. Again by Claim 1, \( |V(P_i)| \geq 1 + \min\{2i, 2(k - i)\} \).

Hence

\[
|V(G)| > |V(D)| \geq (1 + 3 + \ldots + [(k+1)/2]) + (1 + 3 + \ldots + [(k+1)/2]) \geq \frac{(k+1)^2}{2} > n,
\]
a contradiction. \( \Box \)

Applications.

4.4. **Discharging for planar graphs.**

Examples of discharging, versions of Euler’s Formula. The FCT.

A normal plane map is a connected plane multigraph whose vertex degrees and face lengths all are at least 3.

Lebesgue (1940) proved that each 3-connected plane graph \((G, f)\) with \( \delta(G) \geq 5 \) has a 3-face \((a, b, c)\) with \( d(a) + d(b) + d(c) \leq 19 \). Kotzig (1963) improved 19 to 18 and in 1979 conjectured 17. An example of 17 is obtained from the dodecahedron by inserting a vertex into each face.

**Theorem 4.25** (Borodin, 1989). Every normal plane map \((G, f)\) with \( \delta(G) \geq 5 \) has a 3-face \((a, b, c)\) with \( d(a) + d(b) + d(c) \leq 17 \).

Here Lecture 38 ended.