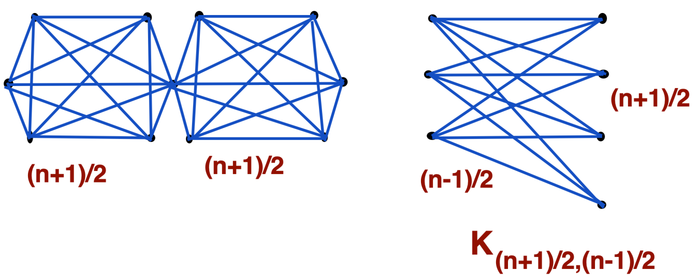
5. Lecture notes: Hamiltonian cycles and circumference

A *Hamiltonian cycle* in a graph G is a cycle passing through all vertices of G. If a graph has a Hamiltonian cycle, then it is also called *Hamiltonian.*

It is an NP-complete problem to check whether a graph has a Hamiltonian cycle. Here are two quite dense graphs with no such cycles.



Theorem 5.1 (Dirac)**.** *Let $n \geq 3$ and G be an n-vertex graph. If $\delta(G) \geq n/2$, then G has a Hamiltonian cycle.*

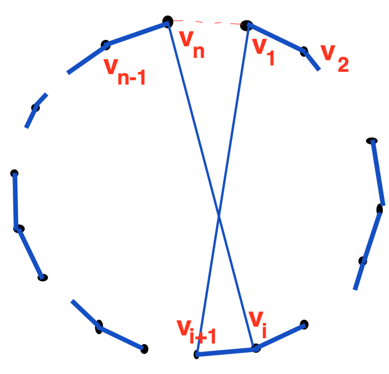
—————————— **Here Lecture 40 ended.**

Proof 1. Suppose the theorem fails for some $n \geq 3$. Let G be an n-vertex simple graph such that

(a) $\delta(G) \geq n/2$,
(b) G has no Hamiltonian cycle, and
(c) G has the most edges among the simple graphs satisfying (a) and (b).

By (b), $G \neq K_n$. Let $xy \notin E(G)$ and $G' = G + xy$. By (c), G' has a Hamiltonian cycle C.

By (b), $xy \in E(C)$. Rename the vertices of G so that $C = v_1, v_2, \ldots, v_n, v_1$, $x = v_1$ and $y = v_n$.



Let $S = N(v_1)$ and $T = \{v_{i+1} : v_i v_n \in E(G)\}$.

If there is $1 \leq i \leq n-1$ s. t. $v_i \in S \cap T$, then G has Hamiltonian cycle $v_1, v_2, \ldots, v_i v_n, v_{n_1}, \ldots, v_{i+1}, v_1$, a contradiction.

Hence S and T are disjoint. Moreover, $v_1 \notin S \cup T$.

So $|S| + |T| + 1 \leq n$. On the other hand, $|S| = d(x) \geq n/2$ and $|T| = d(y) \geq n/2$. So

$$n/2 + n/2 + 1 \leq n,$$

a contradiction. □

Idea of Proof 2 (Original). In the first step, by looking at a longest path, we greedily find a cycle of length at least $1 + n/2$.

In the second step, Dirac considered a lollipop, i.e. a pair (C, P) s.t. C is a cycle and P is a path starting from C.

We maximize $|C|$, and modulo this maximize $|P|$.

The end x of P cannot be adjacent to two consecutive vertices of C, and cannot be adjacent to vertices of C close to the start of P. □

In Proof 1, we actually did not use $\delta(G) \geq n/2$ in full, we needed $|S| + |T| \geq n$.

Let $\sigma_2(G) = \min_{xy \notin E(G)} d(x) + d(y)$. The same Proof 1 gives us

Theorem 5.2 (Ore)**.** *Let $n \geq 3$ and G be an n-vertex simple graph. If $\sigma_2(G) \geq n$, then G has a Hamiltonian cycle.*

A slight modification of Proof 1 gives the following refinement.

Theorem 5.3 (Pósa)**.** *Let $n \geq 3$, $k \geq 0$ and let G be an n-vertex graph with $\sigma_2(G) \geq n + k$. Then for each linear forest $F \subset G$ with k edges G has a Hamiltonian cycle containing F.*

Proof. Suppose the theorem fails for some n. Let G be an n-vertex simple graph such that

(a) $\sigma_2(G) \geq n + k$

(b) for some linear forest $F \subset G$ with k edges, G has no Hamiltonian cycle containing F, and

(c) G has the most edges among the simple graphs satisfying (a) and (b).

By (b), $G \neq K_n$. Let $xy \notin E(G)$ and $G' = G + xy$. By (c), G' has a Hamiltonian cycle C containing F.

By (b), $xy \in E(C)$. Rename the vertices of G so that $C = v_1, v_2, \ldots, v_n, v_1$, $x = v_1$ and $y = v_n$.

Let $S = N(v_1)$ and $T = \{v_{i+1} : v_i v_n \in E(G) \,\text{and}\, v_i v_{i+1} \notin E(F)\}$.

If there is $1 \leq i \leq n-1$ s. t. $v_i \in S \cap T$, then G has Hamiltonian cycle $v_1, v_2, \ldots, v_i v_n, v_{n_1}, \ldots, v_{i+1}, v_1$ containing F, a contradiction.

Therefore, S and T are disjoint. Moreover, $v_1 \notin S \cup T$.

Hence $|S| + |T| + 1 \leq n$. On the other hand, $|S| = d(x) - |E(F)| = d(x) - k$ and $|T| = d(y)$. So $d(x) + d(y) - k + 1 \leq n$, a contradiction. □

Corollary 5.4. *Let $n \geq 3$ and let G be an n-vertex graph with $\sigma_2(G) \geq n + 1$. Then G is hamiltonian-connected.*

Corollary 5.5. *Let $n \geq 3$ and let G be an n-vertex graph. Let $x, y \in V(G)$ and $xy \notin E(G)$. Suppose $d(x) + d(y) \geq n$. Then G has a Hamiltonian cycle iff $G + xy$ has a Hamiltonian cycle.*

Notion of n-closure!

Theorem 5.6 (Chvátal)**.** *Let $n \geq 3$ and let G be a graph with vertex set $V = \{v_1, \ldots, v_n\}$. Suppose that $d(v_1) \leq d(v_2) \leq \ldots \leq d(v_n)$ and for every $i < n/2$*

$$d(v_i) > i \quad \text{or} \quad d(v_{n-i}) \geq n - i. \tag{1}$$

Then G has a Hamiltonian cycle.

Sharpness example: $\overline{K_i}$—K_i—K_{n-2i}.

———————————— **Here Lecture 41 ended.**

Proof. Suppose the theorem fails for some $n \geq 3$. Let G be an n-vertex counter-example with the most edges. By maximality, for each $xy \notin E(G)$ there is a Hamiltonian x, y-path. Choose such a pair (x, y) with maximum $d(x) + d(y)$ and $d(x) \leq d(y)$. By Corollary 5.5,

$$d(x) + d(y) \leq n - 1. \tag{2}$$

Let $t = d(x)$. By (2), $t \leq (n-1)/2$. Let $P = w_1, w_2, \ldots, w_n$ be a Hamiltonian x, y-path (so $x = w_1, y = w_n$) and $w_{i_1}, w_{i_2}, \ldots, w_{i_t}$ be the neighbors of x on P (picture in class!). Clearly, $i_1 = 2$. By the choice of (x, y), $d(w_{i_j - 1}) \leq t$ for all $1 \leq j \leq t$.

Hence $d(v_t) \leq t$, and each of $n - t - 1$ nonneighbors of x has degree $\leq d(y)$. Together with x, at least $n - t$ vertices have degree $\leq d(y) \leq n - 1 - t$. This yields $d(v_{n-t}) \leq n - t - 1$, which together with $d(v_t) \leq t$ contradicts (1). $\square$

Theorem 5.7 (Chvátal and Erdős)**.** *Let $n \geq 3$ and let G be an n-vertex graph. If $\kappa(G) \geq \alpha(G)$, then G has a Hamiltonian cycle.*

Sharpness example: $K_{r,r+1}$.

Proof. Suppose $k = \kappa(G)$ and $C = v_1, \ldots, v_t$ is a longest cycle in G. Let H be a component of $G - V(C)$ and let $S = \{v_{i_1}, \ldots, v_{i_s}\} = N(H) \cap V(C)$. Note that S does not contain consecutive vertices on C, and so $s \leq t/2$. Since G is k-connected, $s \geq k$.

Let $S^- = \{v_{j-1} : v_j \in S\}$. If two vertices in S^- are adjacent, then C is not a longest path (picture in class!!). Thus, for any $v \in V(H)$, the set $S^- + v$ is independent, implying $\alpha(G) \geq 1 + |S^-| \geq 1 + s \geq 1 + k$, a contradiction. $\square$

Circumference, $c(G)$, the number of vertices in the longest path, $p(G)$.

Dirac observed that

$$c(G) \geq \delta(G) + 1 \text{ for every graph } G. \tag{3}$$

We have used this.

Theorem 5.8 (Erdős and Gallai, 1959)**.** *Let $n \geq 3$, $k \geq 2$ and let G be an n-vertex graph.*

(A) If $|E(G)| > (k-1)(n-1)/2$, then $c(G) \geq k$.

(B) If $|E(G)| > (k-2)n/2$, then $p(G) \geq k$.

Sharpness examples: For (B): many disjoint copies of K_{k-1}. For (A): Many copies of K_{k-1} all sharing the same vertex.

Let $n \geq k$, $\frac{k}{2} > a \geq 1$. Define the n-vertex $H_{n,k,a}$:
$V(H_{n,k,a}) = A \cup B \cup C$, where $|A| = a$, $|B| = n - k + a$, $|C| = k - 2a$.
$H_{n,k,a}[A \cup C] = K_{k-a}$, $H_{n,k,a}[A \cup B] = K_{n-k+2a} - E(K_{n-k+a})$ and no edges between B and C.

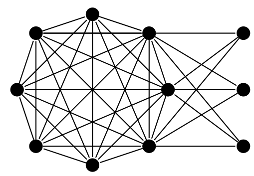


FIGURE 1. Graph $H_{11,11,3}$.

Graph $H_{n,k,a}$ has no cycles of length k or greater.
Let
$$h(n,k,a) = e(H_{n,k,a}) = \binom{k-a}{2} + a(n-k+a).$$

Theorem 5.9 (Kopylov, 1977). *Let $n \geq k \geq 5$ and $t = \lfloor \frac{k-1}{2} \rfloor$. If G is an n-vertex 2-connected graph with no cycle of length at least k, then*

$$e(G) \leq \max\{h(n,k,2), h(n,k,t)\}. \tag{4}$$

Lemma 5.10 (Kopylov, 1977). *Let $n \geq 3$ and G be a 2-connected n-vertex graph. Let $x, y \in V(G)$ and P be an x,y-path with m edges. Then*

$$c(G) \geq \min\{m+1, d_{G[P]}(x) + d_{G[P]}(y)\}. \tag{5}$$

———————————— **Here Lecture 42 ended.**

5.1. **Proof of Theorem 5.9 modulo Lemma 5.10.** Among the counterexamples with n vertices, choose a graph G with the most edges. Then G has the following properties:

(A1) $e(G) > h(n,k,t) = \binom{k-t}{2} + t(n-k+t)$ and $e(G) > h(n,k,2) = \binom{k-2}{2} + 2(n-k+2)$;

(A2) $c(G) \leq k-1$; and

(A3) for each pair $\{x,y\}$ of the vertices in G with $xy \notin E(G)$, G has an x,y-path $P_{x,y}$ with at least $k-1$ edges.

A *j-disintegration of* G is a consecutive deletion of vertices of degree at most j while it is possible.

(6) *If j-disintegration of H deletes all its edges, then* $e(H) \leq \binom{j+1}{2} + j(|V(H)| - j - 1)$.

Let G' be obtained from G by t-disintegration. If $G' = \emptyset$, then by (6),
$$e(G) \leq \binom{t+1}{2} + t(n-t-1) = h(n,k,t),$$
a contradiction to (A1).

So, G' is a non-empty graph with $\delta(G') \geq t+1$. Let $L = V(G')$ and $\ell = |L|$.

Claim 5.11. *$G' = K_\ell$.*

Proof. Suppose $x, y \in V(G')$ and $xy \notin E(G')$. By (A3), G has a path $P_{x,y}$ of length at least $k-1$. Let P be a longest path in G with both ends in L and let u and v be its ends. Then the length of P is at least $k-1$. Since P is longest, neither u nor v has neighbors in $L - V(P)$. Thus $d_{G[P]}(u) \geq d_{G'}(u) \geq t+1$ and $d_{G[P]}(v) \geq d_{G'}(v) \geq t+1$. So by Lemma 5.10,

$$c(G) \geq \min\{(k-1)+1, (t+1)+(t+1)\} = \min\{k, 2t+2\} = k,$$

a contradiction to (A2). $\square$

Observe that $t+1 \leq \ell \leq k-2$. Let G'' be obtained from G by $(k-\ell)$-disintegration, let $V(G'') = M$ and let $m = |M|$. Certainly, $G'' \supseteq G'$. If $G'' = G'$, then

$$e(G) \leq \binom{\ell}{2} + (k-\ell)(n-k+\ell) = h(n,k,k-\ell),$$

a contradiction to (A1). Thus $M - L \neq \emptyset$.

Claim 5.12. *Every $x \in M - L$ is adjacent to each $y \in L$.*

Proof. Suppose $x \in M - L$, $y \in L$ and $xy \notin E(G')$. By (A3), G has a path $P_{x,y}$ of length at least $k-1$. Let P be a longest path in G with both ends in M and at least one end in L. Let u and v be its ends such that $v \in L$. Then the length of P is at least $k-1$. Since P is longest, neither u nor v has neighbors in $L - V(P)$. Thus $d_{G[P]}(u) \geq d_{G'}(u) \geq (k-\ell)+1$ and $d_{G[P]}(v) \geq d_{G'}(v) \geq \ell - 1$. So by Lemma 5.10,

$$c(G) \geq \min\{(k-1)+1, (k-\ell+1)+(\ell-1)\} = \min\{k,k\} = k,$$

a contradiction to (A2). $\square$

But then $\delta(G'') \geq \ell$, and G' cannot be the result of t-disintegration of G. $\square$

REVIEW OF THE COURSE.